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Energy Decay for a Damped Nonlinear Hyperbolic Equation

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Abstract—This paper proves uniform stabilization of the energy of a nonlinear damped hyperbolic equation. The idea of the proof is the use of a specific integral inequality. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper, we are concerned with the energy decay of the solution to the initial boundary value problem for the nonlinear damped hyperbolic equation

$$u'' + k_1\Delta^2 u + k_2\Delta^2 u' + \Delta\, g(\Delta u) = 0, \qquad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = 0, \qquad \text{on } \Gamma \times \mathbb{R}^+, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \Gamma \times \mathbb{R}^+, \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u'(x,0) = u_1(x), \qquad \text{on } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ, k_1 and k_2 are two positive constants, and g is C^2-class real valued function.

This problem describes the motion of the neo-Hookean elastomer rod; for more physical interpretation of problems (1.1)–(1.4) we refer to [1].

This problem has attracted much attention in recent years; for the well-posedness we refer the reader to [2–6]. Quite recently, Banks *et al.* [1] have been successful in proving the global existence of weak solutions by using a variational approach and the semigroup formulation.

In [1], however, no result is given concerning the decay property of solutions and it is desirable to establish the uniform stabilization of solutions to (1.1)–(1.4).

The function spaces we use are all standard and the definitions of them are omitted.

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The following lemma is useful in deriving decay rates of solutions.

LEMMA 1.1. *(See [7, Theorem 8.1].) Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and assume that there exists a constant $T > 0$ such that*

$$\int_S^\infty E(t) \leq TE(S), \qquad \forall S \in \mathbb{R}^+,$$

then

$$E(t) \leq E(0)\, e^{1-t/T}, \qquad \forall t \geq 0.$$

Now the following existence theorem is proved in [1].

THEOREM 1.2. *Let (u_0, u_1) belong to $H_0^2(\Omega) \times L^2(\Omega)$. Assume that*

there exist positive constants c_i for $i = 1, 2, 3$, such that

$$\frac{-1}{2}(k_1 + k_2 - \epsilon)\,|x|^2 - c_1 \leq G(x) \leq c_2\,|x|^2 + c_3, \tag{H1}$$

for $\epsilon > 0$, where we set $G(x) = \int_0^x g(t)\,dt$.

There are positive constants c_i for $i = 1, 2$, such that

$$|g(x)| \leq \tilde{c_1}\,|x| + \tilde{c_2}. \tag{H2}$$

$$g'(x) \geq -a, \qquad \text{for } a > 0. \tag{H3}$$

Then (1.1)–(1.4) admits a unique solution

$$u \in C\left(\mathbb{R}^+;\ H_0^2(\Omega)\right) \cap C^1\left(\mathbb{R}^+;\ L^2(\Omega)\right).$$

We define the energy of (1.1)–(1.4) at time t by the following formula:

$$E(t) = \frac{1}{2}\int_\Omega \left(|u'|^2 + k_1\,|\Delta\, u|^2\right)\,dx + \int_\Omega G(\Delta\, u)\,dx.$$

A simple computation gives

$$E(S) - E(T) = k_2 \int_S^T \int_\Omega |\Delta u'|^2\,dx, \qquad \text{for all } 0 \leq S < T < \infty,$$

for any regular solution of (1.1)–(1.4). This identity remains valid for all mild solution by an easy density argument. So the energy is nonincreasing, and our main result is as follows.

MAIN THEOREM. *Let hypotheses (H1)–(H3) on g be valid and let u be a solution of problems (1.1)–(1.4) in the class*

$$C\left(\mathbb{R}^+;\ H_0^2(\Omega)\right) \cap C^1\left(\mathbb{R}^+;\ L^2(\Omega)\right),$$

then under the assumptions

$$xg(x) \geq 0, \qquad \text{for all } x \in \mathbb{R}, \tag{H4}$$

$$2G(x) \leq xg(x), \qquad \text{for all } x \in \mathbb{R}, \tag{H5}$$

we have

$$E(t) \leq E(0)\, e^{1-\omega t}, \qquad \forall t \in \mathbb{R}^+,$$

where $1/\omega = c(\Omega)/k_2 + \max\{1, (c(\Omega)/k_1)\} + k_2/k_1$ and $c(\Omega)$ is the constant appearing in the following inequality:

$$\int_\Omega u^2\,dx \leq c(\Omega) \int_\Omega |\Delta u|^2\,dx, \qquad \text{for all } u \in H_0^2(\Omega).$$

2. PROOF OF THE MAIN THEOREM

Applying a density argument, it is sufficient to consider the case where u_0 and u_1 are sufficiently smooth to justify all the computations that follow.

Multiplying (1.1) with u, we have

$$\int_S^T \int_\Omega \left(k_1 |\Delta u|^2 - |u'|^2 + 2G(\Delta u)\right) dx\, dt$$
$$= - \left[\int_\Omega uu'\, dx\right]_S^T - \frac{k_2}{2}\left[\int_\Omega |\Delta u|^2\, dx\right]_S^T + \int_\Omega (2G(\Delta u) - \Delta u g(\Delta u))\, dx\, dt,$$

with $0 \le S < T < \infty$.

Whence,

$$2\int_S^T E(t)\, dt = 2\int_S^T \int_\Omega |u'|^2\, dx\, dt$$
$$- \left[\int_\Omega uu'\, dx\right]_S^T - \frac{k_2}{2}\left[\int_\Omega |\Delta u|^2\, dx\right]_S^T + \int_S^T \int_\Omega (2G(\Delta u) - \Delta u g(\Delta u))\, dx\, dt.$$

That is, by (H5)

$$2\int_S^T E(t)\, dt \le 2\int_S^T \int_\Omega |u'|^2\, dx - \left[\int_\Omega uu'\, dx\right]_S^T - \frac{k_2}{2}\left[\int_\Omega |\Delta u|^2\, dx\right]_S^T. \tag{2.1}$$

Using the nonincreasing property of E, the Cauchy-Schwarz inequality, and the definition of E, we have

$$2\int_S^T \int_\Omega |u'|^2\, dx\, dt \le 2c(\Omega)\int_S^T \int_\Omega |\Delta u'|^2\, dx\, dt$$
$$= \frac{2c(\Omega)}{k_2}\int_S^T -E'(t)\, dt \le \frac{2c(\Omega)}{k_2} E(S);$$
$$\left|\int_\Omega uu'\, dx\right| \le \frac{1}{2}\int_\Omega \left(|u|^2 + |u'|^2\right) dx \le \frac{1}{2}\int_\Omega \left(c(\Omega)|\Delta u|^2 + |u'|^2\right) dx$$
$$\le \max\left\{1, \frac{c(\Omega)}{k_1}\right\} E(t) \le \max\left\{1, \frac{c(\Omega)}{k_1}\right\} E(S);$$
$$\frac{k_2}{2}\left|\int_\Omega |\Delta u|^2\, dx\right| \le \frac{k_2}{k_1}E(t) \le \frac{k_2}{k_1}E(S).$$

Using these estimates, we conclude from (2.1) that

$$2\int_S^T E(t)\, dt \le \frac{2c(\Omega)}{k_2}E(S) + 2\max\left\{1, \frac{c(\Omega)}{k_1}\right\} E(S) + \frac{2k_2}{k_1}E(S),$$

and then

$$\int_S^T E(t)\, dt \le \left(\frac{c(\Omega)}{k_2} + \frac{k_2}{k_1} + \max\left\{1, \frac{c(\Omega)}{k_1}\right\}\right) E(S), \qquad \text{for all } 0 \le S < T < +\infty.$$

Letting $T \to +\infty$, this yields the following estimate:

$$\int_S^\infty E(t)\, dt \le \left(\frac{c(\Omega)}{k_2} + \frac{k_2}{k_1} + \max\left\{1, \frac{c(\Omega)}{k_1}\right\}\right) E(S), \qquad \forall\, S \ge 0,$$

and we conclude from Lemma 1.1 that

$$E(t) \le E(0)\, e^{1-\omega t}, \qquad \forall\, t \ge 0,$$

with $1/\omega = c(\Omega)/k_2 + k_2/k_1 + \max\{1, (c(\Omega)/k_2)\}$.

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