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On the stabilization of Timoshenko systems with memory cand different speeds of wave propagation

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ABSTRACT

In this work we consider a one-dimensional Timoshenko system with different speeds of wave propagation and with only one control given by a viscoelastic term on the angular rotation equation. For a wide class of relaxation functions and for sufficiently regular initial data, we establish a general decay result for the energy of solution. Unlike the past history and internal feedback cases, the second energy is not necessarily decreasing. To overcome this difficulty, a precise estimate of the second energy, in terms of the initial data and the relaxation function, is proved.

1. Introduction

In the present work we are concerned with the asymptotic behavior of the solution of the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x = 0 & \text{in } \;]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) + \int_0^t g(t-s)\psi_{xx}(s)ds = 0 & \text{in } \;]0,L[\times\mathbb{R}_+, \\ \varphi(0,t)=\psi(0,t)=\varphi(L,t)=\psi(L,t)=0 & \text{in } \; \mathbb{R}_+, \\ \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x) & \text{on } \;]0,L[, \\ \psi(x,0)=\psi_0(x), \quad \psi_t(x,0)=\psi_1(x) & \text{on } \;]0,L[, \end{cases} \tag{P}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, φ is the transverse displacement of the beam, ψ is the rotation angle of the filament of the beam, $g:\mathbb{R}_+\to\mathbb{R}_+$ is a non-increasing function, and the coefficients ρ_1,ρ_2,k_1 and k_2 are positive constants denoting, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, the shear modulus and Young's modulus of elasticity times the moment of inertia of a cross section and satisfying

$$\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}. \tag{1.1}$$

Our aim is to establish a general decay result, depending on g, for the energy of the system (P).

The Timoshenko system which describes the transverse vibration of a beam was first introduced in [24] and has the form

$$\begin{cases} \rho_1\varphi_{tt} = k_1(\varphi_x-\psi)_x & \text{in } \;]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} = k_2\psi_{xx} + k_1(\varphi_x-\psi) & \text{in } \;]0,L[\times\mathbb{R}_+. \end{cases} \tag{1.2}$$

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Since then many people have been interested in the question of stability of (1.2) with different kind of controls: internal, boundary feedback, memory or past history. Let us mention some of these results.

If both the rotation angle and the transverse displacement are controlled, then it is well known that (1.2) is stable for any weak solution and without any restriction on the constants ρ_1, ρ_2, k_1 and k_2. Many decay estimates were obtained in this case; see for example [3,7–10,12,16,21,22,25–27].

If only the rotation angle is controlled, then there are two different cases. The case of different wave speed of propagation (1.1) and the opposite case. For the case $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$, it is well known that, similarly to the case of two controls, (1.2) is stable and similar decay results were obtained. We quote in this regard [2,4,5,11,13–15,17–20,23]. If (1.1) holds (which is more interesting from the physics point of view), then it is well known that (1.2) is not exponentially stable even for exponentially decaying relaxation functions. Moreover, some polynomial decay estimates for the strong solution of (1.2) were established only for the case of internal feedback in [1] and the case of past history in [15,20]. In these papers, the idea of the proof of the polynomial decay results exploits the non-increasingness property of the second energy (the energy of the system resulting from differentiating the original system with respect to time) to estimate some higher-order terms.

In the case of memory control (P), the second energy is not necessarily non-increasing. To overcome this difficulty, we give an explicit estimate for the second energy in terms of the relaxation function and the initial data. In addition, we consider here a wider class of relaxation functions g than those considered in the case of past history control [15,20].

The paper is organized as follows. In Section 2, we state some hypotheses and present our stability result. In Section 3, we give the proof of our stability result.

2. Preliminaries

We consider the following hypothesis:(H) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function satisfying

$$g(0) > 0, \quad k_2 - \int_0^{+\infty} g(s)ds =: l > 0 \tag{2.1}$$

and there exists a non-increasing differentiable function ξ: $\mathbb{R}_+ \to]0, +\infty[$ and a constant $p \geqslant 1$ such that

$$g'(t) \leqslant -\xi(t) g^p(t), \quad \forall t \geqslant 0. \tag{2.2}$$

Remark 2.1. Condition (2.2) describes better the growth of g at infinity and allows us to obtain precise estimate of the energy and more general than the "stronger" one ($\xi =$ constant and $p \in [1, \frac{3}{2}[$) used in the case of past history control [15,20]. We consider here the form (2.2) because our decay estimate can be expressed in a better way in the case $\xi =$ constant, than in the one $p = 1$.

Remark 2.2. By using a standard Galerkin method, we can show that (P) has, for any initial data

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in \left(H^2(]0, L[) \cap H_0^1(]0, L[)\right) \times H_0^1(]0, L[),$$

a unique (strong) solution

$$\varphi, \psi \in C\left(\mathbb{R}^+; H^2(]0, L[) \cap H_0^1(]0, L[)\right), \tag{2.3}$$

$$\cap C^1\left(\mathbb{R}^+; H_0^1(]0, L[)\right) \cap C^2\left(\mathbb{R}^+; L^2(]0, L[)\right),$$

and for any initial data

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(]0, L[) \times L^2(]0, L[),$$

problem (P) has a unique (weak) solution

$$\varphi, \psi \in C\left(\mathbb{R}^+; H_0^1(]0, L[)\right) \cap C^1\left(\mathbb{R}^+; L^2(]0, L[)\right). \tag{2.4}$$

Now we introduce the energy functional associated with (P) by

$$E(t) := \frac{1}{2} g \circ \psi_x + \frac{1}{2} \int_0^L \left[\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \left(k_2 - \int_0^t g(s)ds\right)\psi_x^2 + k_1(\varphi_x + \psi)^2\right] dx, \tag{2.5}$$

where, for all $v : \mathbb{R}_+ \to L^2(]0, L[)$,

$$g \circ v = \int_0^L \int_0^t g(t-s)(v(t) - v(s))^2 ds dx. \tag{2.6}$$

Our main stability result reads:

Theorem 2.1. *Assume that (H) holds and let*

$$(\varphi_0,\varphi_1),(\psi_0,\psi_1)\in\left(H^2(]0,L[)\cap H_0^1(]0,L[)\right)\times H_0^1(]0,L[).$$

Then the (strong) solution (2.3) satisfies

$$E(t)\leqslant C\left(\frac{1+\left(\int_0^t g^{\frac{1}{2}}(s)ds\right)^{2(p-1)}+\int_0^t g^{2-p}(s)ds}{\int_0^t \xi(s)ds}\right)^{\frac{1}{2p-1}},\quad \forall t>0, \tag{2.7}$$

where C is a constant depending continuously on

$$\|(\varphi_0,\varphi_1)\|^2_{H^2(]0,L[)\times H_0^1(]0,L[)}+\|(\psi_0,\psi_1)\|^2_{H^2(]0,L[)\times H_0^1(]0,L[)}.$$

Remark 2.3. 1. If $\frac{-g'}{g}$ is differentiable and non-increasing, then (2.2) is satisfied with $\xi=\frac{-g'}{g}$ and $p=1$. Consequently, we have at least the estimate

$$E(t)\leqslant C\Big/\ln\frac{g(0)}{g(t)},\quad \forall t>0. \tag{2.8}$$

2. If $\xi=$ constant and $1\leqslant p<\frac{3}{2}$ (hence $\int_0^{+\infty} g^{\frac{1}{2}}(s)ds<+\infty$ and $\int_0^{+\infty} g^{2-p}(s)ds<+\infty$), then (2.7) becomes

   $$E(t)\leqslant Ct^{\frac{-1}{2p-1}},\quad \forall t>0.$$

3. The best decay rate given by (2.7) is $E(t)\leqslant\frac{C}{t}$ which holds when $\xi=$ constant and $p=1$ (that is; g decays exponentially to zero).

Examples. Let us give here some examples to illustrate our general estimate (2.7) and the difference between the cases $\xi=$ constant and $p=1$.

1. Let $g(t)=\frac{a}{(2+t)(\ln(2+t))^q}$, for $q>1$, and let $a>0$ be small enough so that (2.1) is satisfied. Condition (2.2) is satisfied with $\xi=\frac{-g'}{g}$ and $p=1$ (but it is not satisfied with $\xi=$ constant and $p\in]1,2[$) and then (2.8) gives

   $$E(t)\leqslant\frac{C}{\ln(t+2)+\ln(\ln(t+2))},\quad \forall t\geqslant 0.$$

2. Let $g(t)=\frac{a}{(1+t)^q}$, for $q>1$, and let $a>0$ be small enough so that (2.1) is satisfied. Condition (2.2) is satisfied with $\xi=$ constant and $p=1+\frac{1}{q}$. Then (2.7) gives

   $$E(t)\leqslant\begin{cases}Ct^{\frac{-q(q-1)}{q+2}} & \text{if}\quad q\in]1,2[\\ Ct^{\frac{-q}{q+2}} & \text{if}\quad q\in]2,+\infty[,\\ C(\ln(1+t))^{1/2}t^{-1/2} & \text{if}\quad q=2,\end{cases}\quad \forall t>0.$$

3. Let $g(t)=ae^{-(\ln(1+t))^q}$, for $q>1$, and let $a>0$ be small enough so that (2.1) is satisfied. Condition (2.2) is satisfied with $\xi=$ constant and any $p>1$. Then (2.7) gives, for any $r<1$,

   $$E(t)\leqslant Ct^{-r},\quad \forall t>0.$$

Here (2.2) is also satisfied with $\xi=\frac{-g'}{g}$ and $p=1$, and (2.8) gives the "weaker" estimate

$$E(t)\leqslant C(\ln(t+1))^{-q},\quad \forall t>0.$$

4. Let $g(t)=ae^{-(1+t)^q}$, for $q>0$, and let $a>0$ be small enough so that (2.1) is satisfied. Condition (2.2) is satisfied with $\xi=$ constant and $p=1$ if $q\geqslant 1$, and it is satisfied with $\xi=$ constant and any $p>1$ if $q\in]0,1[$. Then (2.7) gives

   $$E(t)\leqslant\begin{cases}Ct^{-1} & \text{if}\quad q\geqslant 1,\\ Ct^{-r} & \text{if}\quad q\in]0,1[(\text{for any } r<1)\end{cases},\quad \forall t>0.$$

If $q\in]0,1[$, then (2.2) is also satisfied with $\xi=\frac{-g'}{g}$ and $p=1$, and the estimate (2.8) gives the "weaker" estimate

$$E(t)\leqslant C(t+1)^{-q},\quad \forall t\geqslant 0.$$

Remark 2.4. 1. The examples above show that the function g satisfying (2.2) with $\xi =$ constant leads to a better decay estimate than the one obtained for those satisfying (2.2) with $p = 1$. But the presence of function ξ in (2.2) allows a wider class of relaxation functions g. In the case of past history control [15,20], condition (2.2) was considered with $\xi =$ constant and $p \in [1, \frac{3}{2}[$ only. The more general growth of g, for the case of infinite history, has been discussed lately by Guesmia [6] in an abstract setting.

2. The case of equal wave speed of propagation $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$, condition (2.2) was considered with $\xi =$ constant and $p \in [1, \frac{3}{2}[$ in [4,6], and with $p = 1$ in [5,13]. The obtained results, in those papers, show that the energy of (P) obeys (for positive constants c_1, c_2)

$$E(t) \leqslant \begin{cases} c_1 e^{-c_2 t} & \text{if} \quad p = 1 \quad \text{and} \quad \xi = \text{constant} \\ c_1 (1+t)^{-\frac{1}{p-1}} & \text{if} \quad p \in]1, \frac{3}{2}[\quad \text{and} \quad \xi = \text{constant} \end{cases} \tag{2.9}$$

and, in general,

$$E(t) \leqslant c_1 e^{-\int_0^t \xi(s)ds}, \quad \text{if} \quad p = 1. \tag{2.10}$$

Here (2.10) is better than (2.9) in general (see examples above) and it is useless to consider $p > 1$. As conclusion of these two remarks, it seems that condition (2.2) is more appropriate for the case $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$ when $p = 1$, and to the case (1.1) when $\xi =$ constant.

3. Proof of the main result

In this section we prove our main stability result. For this purpose we establish several lemmas. We will use c (sometimes c_τ which depends on some parameter τ), throughout this paper, to denote a generic positive constant which does not depend on the initial data.

Lemma 3.1. *Let (φ, ψ) be the strong solution of (P). Then the energy functional satisfies, for any $t \geqslant 0$,*

$$E'(t) = -\frac{1}{2} g(t) \int_0^L \psi_x^2 dx + \frac{1}{2} g' \circ \psi_x \leqslant 0. \tag{3.1}$$

Proof. By multiplying the first two equations in (P) by φ_t and ψ_t respectively, integrating over $]0, L[$ and using boundary conditions, we obtain (3.1).

Let us set

$$g_1(t) = \left(\int_0^t g^{\frac{1}{2}}(s)ds \right)^{2(p-1)}, \quad g_2(t) = \int_0^t g^{2-p}(s)ds. \qquad \square$$

Lemma 3.2. *Let (φ, ψ) be the strong solution of (P). Then*

$$(g \circ \psi_x)^{2p-1} \leqslant \left(\frac{8}{l} E(0) \right)^{2(p-1)} g_1(t) g^p \circ \psi_x, \tag{3.2}$$

$$\left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s)ds \right)^2 \leqslant g_2(t) \int_0^t g^p(t-s)(\psi_x(t) - \psi_x(s))^2 ds, \tag{3.3}$$

$$\left(\int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))ds \right)^2 \leqslant -g(0) \int_0^t g'(t-s)(\psi_x(t) - \psi_x(s))^2 ds. \tag{3.4}$$

Proof.

1. If $p = 1$, then (3.2) is trivial.
   If $p > 1$, let $r = \frac{2p-1}{2(p-1)}$ and use Hölder's inequality to obtain

$$\begin{aligned}\int_0^L\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))^2dsdx &= \int_0^L\int_0^t\left[g^{\frac{1}{2r}}(t-s)(\psi_x(t)-\psi_x(s))^{\frac{2}{r}}\right]\left[g^{\frac{2r-1}{2r}}(t-s)(\psi_x(t)-\psi_x(s))^{\frac{2(r-1)}{r}}\right]dsdx\\ &\leqslant \left(\int_0^L\int_0^t g^{\frac{1}{2}}(t-s)(\psi_x(t)-\psi_x(s))^2dsdx\right)^{\frac{1}{r}}\times\left(\int_0^L\int_0^t g^{\frac{2r-1}{2(r-1)}}(t-s)(\psi_x(t)-\psi_x(s))^2dsdx\right)^{\frac{r-1}{r}}\\ &\leqslant \left(2\int_0^t g^{\frac{1}{2}}(t-s)\int_0^L(\psi_x^2(t)+\psi_x^2(s))dxds\right)^{\frac{2(p-1)}{2p-1}}\times\left(\int_0^L\int_0^t g^p(t-s)(\psi_x(t)-\psi_x(s))^2dsdx\right)^{\frac{1}{2p-1}}.\end{aligned}$$

Using the definition and the non-increasingness of E, we get

$$\left(2\int_0^t g^{\frac{1}{2}}(t-s)\int_0^L(\psi_x^2(t)+\psi_x^2(s))dxds\right)^{\frac{2(p-1)}{2p-1}}\leqslant\left(\frac{4}{l}\int_0^t g^{\frac{1}{2}}(t-s)(E(t)+E(s))ds\right)^{\frac{2(p-1)}{2p-1}}\leqslant\left(\frac{8}{l}E(0)\int_0^t g^{\frac{1}{2}}(s)ds\right)^{\frac{2(p-1)}{2p-1}}=\left(\frac{8}{l}E(0)\right)^{\frac{2(p-1)}{2p-1}}(g_1(t))^{\frac{1}{2p-1}}.$$

Therefore (3.2) follows.

2. Similarly, using Cauchy–Schwarz' inequality, we get

$$\begin{aligned}\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 &= \left(\int_0^t[g^{1-\frac{p}{2}}(t-s)][g^p(t-s)(\psi_x(t)-\psi_x(s))^2ds\right)\\ &\leqslant\left(\int_0^t g^{2-p}(t-s)ds\right)\left(\int_0^t g^p(t-s)(\psi_x(t)-\psi_x(s))^2ds\right)\\ &\leqslant\left(\int_0^t g^{2-p}(s)ds\right)\left(\int_0^t g^p(t-s)(\psi_x(t)-\psi_x(s))^2ds\right)\end{aligned}$$

which gives (3.3).

3. To prove (3.4), we proceed as in case of (3.3):

$$\begin{aligned}\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 &= \left(\int_0^t -g'(t-s)ds\right)\left(\int_0^t -g'(t-s)(\psi_x(t)-\psi_x(s))^2ds\right)\\ &\leqslant -g(0)\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))^2ds;\end{aligned}$$

hence, (3.4) follows. □

Lemma 3.3. *Let (φ,ψ) be the strong solution of (P). Then the functional*

$$I_1 := -\rho_2\int_0^L\psi_t\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx$$

satisfies, for any $\delta>0$,

$$I_1'(t)\leqslant-\rho_2\left(\int_0^t g(s)ds-\delta\right)\int_0^L\psi_t^2dx+\delta\int_0^L(\varphi_x+\psi)^2dx+\delta\int_0^L\psi_x^2dx+c_\delta g_2(t)g^p\circ\psi_x-c_\delta g'\circ\psi_x. \tag{3.5}$$

Proof. Direct computations, using (P), yield

$$\begin{aligned}I_1'(t) &= -\rho_2\int_0^L\psi_t\left(\int_0^t g'(t-s)(\psi(t)-\psi(s))ds+\left(\int_0^t g(s)ds\right)\psi_t\right)dx\\ &\quad-\int_0^L\left(\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right)\left(k_2\psi_{xx}-k_1(\varphi_x+\psi)-\int_0^t g(t-s)\psi_{xx}(s)ds\right)dx\\ &= -\rho_2\int_0^L\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx-\rho_2\left(\int_0^t g(s)ds\right)\int_0^L\psi_t^2dx+k_2\int_0^L\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx\\ &\quad+k_1\int_0^L(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx-\int_0^L\left(\int_0^t g(t-s)\psi_x(s)ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx.\end{aligned}$$

Now we estimate the terms in the right-hand side of the above equality as follows. By using (3.4) and Young's and Poincaré's inequalities we obtain, for any $\delta>0$,

$$-\rho_2\int_0^L\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx\leqslant\delta\rho_2\int_0^L\psi_t^2dx-c_\delta g_2(t)g'\circ\psi_x.$$

Similarly, using (3.3), we have

$$k_2\int_0^L\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx\leqslant\frac{\delta}{2}\int_0^L\psi_x^2dx+c_\delta g_2(t)g^p\circ\psi_x,$$

$$k_1\int_0^L(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx\leqslant\delta\int_0^L(\varphi_x+\psi)^2dx+c_\delta g_2(t)g^p\circ\psi_x$$

and

$$\begin{aligned}&-\int_0^L\left(\int_0^t g(t-s)\psi_x(s)ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx\\&\leqslant\frac{\delta}{4\left(\int_0^{+\infty}g(s)ds\right)^2}\int_0^L\left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t)+\psi_x(t))ds\right)^2dx+c_\delta\int_0^L\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2dx\\&\leqslant\frac{\delta}{4\left(\int_0^{+\infty}g(s)ds\right)^2}\int_0^L 2\left(\int_0^t g(s)ds\right)^2\psi_x^2dx+c_\delta\int_0^L\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2dx\leqslant\frac{\delta}{2}\int_0^L\psi_x^2dx+c_\delta g_2(t)g^p\circ\psi_x.\end{aligned}$$

A combination of all the above estimates yields (3.5). □

Lemma 3.4. *Let (φ,ψ) be the strong solution of (P). Then the functional*

$$I_2(t):=-\int_0^L(\rho_2\psi\psi_t+\rho_1\varphi\varphi_t)dx$$

satisfies the estimate

$$I_2'(t)\leqslant-\int_0^L(\rho_2\psi_t^2+\rho_1\varphi_t^2)dx+k_1\int_0^L(\varphi_x+\psi)^2dx+c\int_0^L\psi_x^2dx+cg_2(t)g^p\circ\psi_x. \tag{3.6}$$

Proof. By exploiting equations of (P) and repeating the same procedure as above, we have

$$\begin{aligned}I_2'(t)&=-\int_0^L(\rho_2\psi_t^2+\rho_1\varphi_t^2)dx-k_1\int_0^L\varphi(\varphi_x+\psi)_xdx-\int_0^L\psi\left(k_2\psi_{xx}-k_1(\varphi_x+\psi)-\int_0^t g(t-s)\psi_{xx}(s)ds\right)dx\\&=-\int_0^L(\rho_2\psi_t^2+\rho_1\varphi_t^2)dx+k_1\int_0^L(\varphi_x+\psi)^2dx+k_2\int_0^L\psi_x^2dx-\int_0^L\psi_x\int_0^t g(t-s)\psi_x(s)dsdx.\end{aligned}$$

To obtain (3.6) we have just to note that, using (3.3) and Young's inequality,

$$\begin{aligned}-\int_0^L\psi_x\int_0^t g(t-s)\psi_x(s)dsdx&=-\int_0^L\psi_x\int_0^t g(t-s)(\psi_x(s)-\psi_x(t)+\psi_x(t))dsdx\\&\leqslant c\int_0^L\psi_x^2dx+c\int_0^L\left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t))ds\right)^2dx\leqslant c\int_0^L\psi_x^2dx+cg_2(t)g^p\circ\psi_x.\end{aligned}$$ □

Lemma 3.5. *Let (φ,ψ) be the strong solution of (P). Then the functional*

$$I_3(t):=\rho_2\int_0^L\psi_t(\varphi_x+\psi)dx+\frac{k_2\rho_1}{k_1}\int_0^L\psi_x\varphi_tdx-\frac{\rho_1}{k_1}\int_0^L\varphi_t\int_0^t g(t-s)\psi_x(s)dsdx$$

satisfies, for any $\epsilon>0$,

$$\begin{aligned}I_3'(t)\leqslant&\frac{1}{2\epsilon}\left(k_2\psi_x(L,t)-\int_0^t g(t-s)\psi_x(L,s)ds\right)^2+\frac{1}{2\epsilon}\left(k_2\psi_x(0,t)-\int_0^t g(t-s)\psi_x(0,s)ds\right)^2\\&+\frac{\epsilon}{2}\left(\varphi_x^2(L,t)+\varphi_x^2(0,t)\right)-k_1\int_0^L(\varphi_x+\psi)^2dx+\rho_2\int_0^L\psi_t^2dx+\epsilon\int_0^L\varphi_t^2dx+\frac{c}{\epsilon}\int_0^L\psi_x^2dx-c_\epsilon g'\circ\psi_x\\&+\left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L\varphi_t\psi_{xt}dx.\end{aligned} \tag{3.7}$$

Proof. By exploiting equations (P) and repeating the same steps, we have

$$\begin{aligned} I_3'(t) &= \rho_2 \int_0^L (\varphi_{xt} + \psi_t)\psi_t dx + \frac{k_2\rho_1}{k_1}\int_0^L \psi_{xt}\varphi_t dx + \int_0^L (\varphi_x + \psi)\left(k_2\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - k_1(\varphi_x+\psi)\right)dx + k_2 \\ &\quad \times \int_0^L \psi_x(\varphi_x+\psi)_x dx - \int_0^L (\varphi_x+\psi)_x\left(\int_0^t g(t-s)\psi_x(s)ds\right)dx - \frac{\rho_1}{k_1}\int_0^L \varphi_t\left(g(0)\psi_x + \int_0^t g'(t-s)\psi_x(s)ds\right)dx \\ &= -k_1\int_0^L (\varphi_x+\psi)^2 dx + \rho_2\int_0^L \psi_t^2 dx + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx + \left[\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x+\psi)\right]_0^L \\ &\quad - \frac{\rho_1}{k_1}g(t)\int_0^L \psi_x\varphi_t dx + \frac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))dsdx. \end{aligned}$$

By using (3.4) and Young's inequality, for the last three terms of this equality, (3.7) follows.

Now, as in [2], we estimate the boundary terms of (3.7).

Lemma 3.6. *Let $m(x) = 2 - \frac{4}{L}x$ and (φ,ψ) be the strong solution of (P). Then, for any $\epsilon > 0$, the functionals*

$$I_4(t) := \rho_2\int_0^L m(x)\psi_t\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx$$

and

$$I_5(t) := \rho_1\int_0^L m(x)\varphi_t\varphi_x dx$$

satisfy

$$\begin{aligned} I_4'(t) &\leqslant -\left(k_2\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)ds\right)^2 - \left(k_2\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2 + \epsilon k_1\int_0^L (\varphi_x+\psi)^2 dx \\ &\quad + c\left(1+\frac{1}{\epsilon}\right)\left(\int_0^L \psi_x^2 dx + g_2(t)g^p\circ\psi_x\right) + c\int_0^L \psi_t^2 dx - cg'\circ\psi_x \end{aligned} \tag{3.8}$$

and

$$I_5'(t) \leqslant -k_1\left(\varphi_x^2(L,t) + \varphi_x^2(0,t)\right) + c\int_0^L (\varphi_t^2 + \varphi_x^2 + \psi_x^2)dx. \tag{3.9}$$

Proof. By noting that $m'(x) = -\frac{4}{L}$ and $m(0) = -m(L) = 2$ and exploiting equations (P), similar calculations as in above give

$$\begin{aligned} I_4'(t) &= \int_0^L m(x)\left(k_2\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - k_1(\varphi_x+\psi)\right)\times\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx + \rho_2 \\ &\quad \times\int_0^L m(x)\psi_t\left(k_2\psi_{xt} - g(0)\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)dx \\ &= -\left(k_2\psi_x(L,t) - \int_0^t g(t-s)\psi_x(L,s)ds\right)^2 - \left(k_2\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)ds\right)^2 + \frac{2}{L} \\ &\quad \times\int_0^L\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 + \frac{2\rho_2k_2}{L}\int_0^L \psi_t^2 dx - k_1\int_0^L m(x)\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x+\psi)dx + \rho_2 \\ &\quad \times\int_0^L m(x)\psi_t\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s)ds\right)dx - \rho_2 g(t)\int_0^L m(x)\psi_x\psi_t dx. \end{aligned}$$

By using (3.3) and Young's inequality, we get

$$\begin{aligned} \frac{2}{L}\int_0^L\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2 dx &= \frac{2}{L}\int_0^L\left(\left(k_2 - \int_0^t g(s)ds\right)\psi_x + \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \\ &\leqslant c\int_0^L \psi_x^2 dx + c\int_0^L\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2 dx \\ &\leqslant c\left(\int_0^L \psi_x^2 dx + g_2(t)g^p\circ\psi_x\right). \end{aligned}$$

Similarly, for any $\epsilon > 0$, we have

$$-k_1\int_0^L m(x)\left(k_2\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x+\psi)dx \leqslant \epsilon k_1\int_0^L(\varphi_x+\psi)^2dx+\frac{c}{\epsilon}\int_0^L\left(k_2\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2dx$$
$$\leqslant \epsilon k_1\int_0^L(\varphi_x+\psi)^2dx+\frac{c}{\epsilon}\left(\int_0^L\psi_x^2dx+g_2(t)g^p\circ\psi_x\right).$$

Also, using (3.4), it is clear that

$$\rho_2\int_0^L m(x)\psi_t\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx-\rho_2 g(t)\int_0^L m(x)\psi_t\psi_x dx$$
$$\leqslant c\int_0^L(\psi_t^2+\psi_x^2)dx+c\int_0^L\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)^2dx \leqslant c\int_0^L(\psi_t^2+\psi_x^2)dx-cg'\circ\psi_x.$$

Combining all the above, we obtain (3.8).

Similarly, we can prove (3.9). Indeed,

$$I_5'(t)=k_1\int_0^L m(x)\varphi_x(\varphi_x+\psi)_x dx+\rho_1\int_0^L m(x)\varphi_t\varphi_{xt}dx$$
$$=\frac{k_1}{2}\left[m(x)\varphi_x^2\right]_0^L+\frac{2k_1}{L}\int_0^L\varphi_x^2dx+\frac{2\rho_1}{L}\int_0^L\varphi_t^2dx+k_1\int_0^L m(x)\varphi_x\psi_x dx.$$

Consequently, using Young's inequality for the last integral of this equality, we get (3.9). □

Lemma 3.7. *Let (φ,ψ) be the strong solution of (P) Then, for any $\epsilon\in\,]0,1[$, the functional*

$$I_6(t):=I_3(t)+\frac{1}{2\epsilon}I_4(t)+\frac{\epsilon}{2k_1}I_5(t)$$

satisfies

$$I_6'(t)\leqslant-\left(\frac{k_1}{2}-c\epsilon\right)\int_0^L(\varphi_x+\psi)^2dx+c\epsilon\int_0^L\varphi_t^2dx+\frac{c}{\epsilon}\int_0^L\psi_x^2dx+\frac{c}{\epsilon^2}\int_0^L\psi_x^2dx+c_\epsilon(g_2(t)g^p\circ\psi_x-g'\circ\psi_x)$$
$$+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_t\psi_{tx}dx. \tag{3.10}$$

Proof. By using Poincaré's inequality, we have

$$\int_0^L\varphi_x^2dx\leqslant 2\int_0^L(\varphi_x+\psi)^2dx+2\int_0^L\psi^2dx\leqslant 2\int_0^L(\varphi_x+\psi)^2dx+c\int_0^L\psi_x^2dx.$$

Then (3.7)–(3.9) imply (3.10). □

Lemma 3.8. *Let (φ,ψ) be the strong solution of (P). Then for any $\epsilon\in\,]0,1[$, the functional*

$$I_7(t):=I_6(t)+\frac{1}{8}I_2(t),$$

satisfies

$$I_7'(t)\leqslant-\frac{k_1}{4}\int_0^L(\varphi_x+\psi)^2dx-\frac{\rho_1}{16}\int_0^L\varphi_t^2dx+c\int_0^L(\varphi_t^2+\psi_x^2)dx+c(g_2(t)g^p\circ\psi_x-g'\circ\psi_x)$$
$$+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_t\psi_{tx}dx. \tag{3.11}$$

Proof. Inequality (3.10), with $\epsilon\in\,]0,1[$ small enough, and inequality (3.6) yield (3.11). □

As in [2,17], we use a function w, given by

$$w(x,t)=-\int_0^x\psi(y,t)dy+\frac{1}{L}\left(\int_0^L\psi(y,t)dy\right)x \tag{3.12}$$

to get a crucial estimate.

Lemma 3.9. *The function w satisfies*

$$\int_0^L w_x^2 dx \leqslant c \int_0^L \psi^2 dx \tag{3.13}$$

and

$$\int_0^L w_t^2 dx \leqslant c \int_0^L \psi_t^2 dx. \tag{3.14}$$

Proof. We have just to calculate w_x and use Hölder's inequality to get (3.13). Applying (3.13) to w_t we get

$$\int_0^L w_{tx}^2 dx \leqslant c \int_0^L \psi_t^2 dx.$$

Then, using Poincaré's inequality (note that $w_t(0,t) = w_t(L,t) = 0$), we arrive at (3.14). □

Lemma 3.10. *Let (φ, ψ) be the strong solution of (P). Then for any $\epsilon \in]0,1[$, the functional*

$$I_8(t) := \int_0^L (\rho_2 \psi_t \psi + \rho_1 w \varphi_t) dx$$

satisfies

$$I_8'(t) \leqslant -\frac{l}{2} \int_0^L \psi_x^2 dx + \frac{c}{\epsilon} \int_0^L \psi_t^2 dx + \epsilon \int_0^L \varphi_t^2 dx + c g_2(t) g^p \circ \psi_x. \tag{3.15}$$

Proof. By using equations (P), integrating by parts and taking into account inequalities (3.13) and (3.14), we arrive at

$$\begin{aligned} I_8'(t) &= \int_0^L (\rho_2 \psi_t^2 + \rho_1 w_t \varphi_t) dx + k_1 \int_0^L w(\varphi_x + \psi)_x dx + \int_0^L \psi \left(k_2 \psi_{xx} - k_1(\varphi_x + \psi) - \int_0^t g(t-s) \psi_{xx}(s) ds \right) dx \\ &= \int_0^L (\rho_2 \psi_t^2 - k_2 \psi_x^2) dx + \left(\int_0^t g(s) ds \right) \int_0^L \psi_x^2 dx - k_1 \int_0^L (\psi + w_x)(\varphi_x + \psi) dx + \rho_1 \int_0^L w_t \varphi_t dx \\ &\quad + \int_0^L \psi_x \int_0^t g(t-s)(\psi_x(s) - \psi_x(t)) ds dx. \end{aligned}$$

By recalling (3.3), (3.12) and (3.14) and the fact that

$$w_x = -\psi + \frac{1}{L} \int_0^L \psi(y,t) dy,$$

we get

$$-k_1 \int_0^L (\psi + w_x)(\varphi_x + \psi) = -\frac{k_1}{L} \left(\int_0^L \psi dx \right) \left([\varphi]_{x=0}^{x=L} + \int_0^L \psi dx \right) = -\frac{k_1}{L} \left(\int_0^L \psi dx \right)^2 \leqslant 0,$$

$$-\left(k_2 - \int_0^t g(s) ds \right) \int_0^L \psi_x^2 dx \leqslant -\left(k_2 - \int_0^{+\infty} g(s) ds \right) \int_0^L \psi_x^2 dx = -l \int_0^L \psi_x^2 dx,$$

$$\rho_1 \int_0^L w_t \varphi_t dx \leqslant \epsilon \int_0^L \varphi_t^2 dx + \frac{c}{\epsilon} \int_0^L w_t^2 dx \leqslant \epsilon \int_0^L \varphi_t^2 dx + \frac{c}{\epsilon} \int_0^L \psi_t^2 dx$$

and

$$\int_0^L \psi_x \int_0^t g(t-s) ds (\psi_x(s) - \psi_x(t)) ds dx \leqslant \frac{1}{2} \int_0^L \psi_x^2 dx + c g_2(t) g^p \circ \psi_x.$$

Consequently, (3.15) follows. □

Now, to estimate the term $(\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt}$ appeared in (3.11), we use the system obtained by differentiating (P) with respect to time t.

Noting that $\psi(x,0) = \psi_0(x)$ and using the fact that

$$\frac{\partial}{\partial t} \left[\int_0^t g(t-s) \psi_{xx}(s) ds \right] = \frac{\partial}{\partial t} \left[\int_0^t g(s) \psi_{xx}(t-s) ds \right]$$

$$= \int_0^t g(s)\psi_{xxt}(t-s)ds + g(t)\psi_{xx}(x,0) = \int_0^t g(t-s)\psi_{xxt}(s)ds + g(t)\psi_{0xx},$$

we have

$$\begin{cases} \rho_1\varphi_{ttt} - k_1(\varphi_x+\psi)_{xt} = 0 & \text{in } \;]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{ttt} - k_2\psi_{txx} + k_1(\varphi_{tx}+\psi_t) & \\ + \int_0^t g(t-s)\psi_{txx}(s)ds + g(t)\psi_{0xx} = 0 & \text{in}]\;0,L[\times\mathbb{R}_+, \\ \varphi_t(0,t) = \psi_t(0,t) = \varphi_t(L,t) = \psi_t(L,t) = 0 & \text{on } \;\mathbb{R}_+. \end{cases} \qquad (P_*).$$

Lemma 3.11. *Let (φ,ψ) be the strong solution of (P). Then the energy of (P_*), defined by*

$$E_*(t) := \frac{1}{2}g\circ\psi_{xt} + \frac{1}{2}\int_0^L \left[\rho_1\varphi_{tt}^2 + \rho_2\psi_{tt}^2 + \left(k_2 - \int_0^t g(s)ds\right)\psi_{xt}^2 + k_1(\varphi_{xt}+\psi_t)^2dx\right] \tag{3.16}$$

satisfies, for all $t \geqslant 0$,

$$E_*'(t) = -\frac{1}{2}g(t)\int_0^L \psi_{xt}^2dx + \frac{1}{2}g'\circ\psi_{xt} - g(t)\int_0^L \psi_{tt}\psi_{0xx}dx \tag{3.17}$$

and

$$E_*(t) \leqslant c\left(E_*(0) + \int_0^L \psi_{0xx}^2dx\right). \tag{3.18}$$

Proof. Multiplying the first two equations in (P_*) by φ_{tt} and ψ_{tt}, respectively, integrating over $]0,L[$ and using the boundary conditions, we obtain (3.17). Then we have

$$E_*'(t) \leqslant \frac{1}{2}g(t)\int_0^L\left(\rho_2\psi_{tt}^2 + \frac{1}{\rho_2}\psi_{0xx}^2\right)dx \leqslant g(t)E_*(t) + \frac{1}{2\rho_2}g(t)\int_0^L \psi_{0xx}^2dx,$$

which implies

$$\frac{\partial}{\partial t}\left(E_*(t)e^{-\int_0^t g(s)ds}\right) \leqslant \frac{1}{2\rho_2}e^{-\int_0^t g(s)ds}g(t)\int_0^L \psi_{0xx}^2dx \leqslant \frac{1}{2\rho_2}g(t)\int_0^L \psi_{0xx}^2dx.$$

Then, a simple integration yields

$$E_*(t)e^{-\int_0^{+\infty} g(s)ds} \leqslant E_*(t)e^{-\int_0^t g(s)ds} \leqslant E_*(0) + \frac{1}{2\rho_2}\left(\int_0^t g(s)ds\right)\int_0^L \psi_{0xx}^2dx \leqslant E_*(0) + \frac{1}{2\rho_2}\left(\int_0^{+\infty} g(s)ds\right)\int_0^L \psi_{0xx}^2dx.$$

Hence (3.18) follows. □

Now, let $t_0 > 0$ and $N_1, N_2, N_3 > 0$. We put $g_0 = \int_0^{t_0} g(s)ds$ and

$$I_9(t) := N_1(E(t) + E_*(t)) + N_2I_1(t) + N_3I_8 + I_7(t).$$

By combining (3.1), (3.5), (3.11), (3.15) and (3.17) and taking $\delta = \frac{k_1}{8N_2}$ in (3.5), we obtain, for all $t \geqslant t_0$.

$$I_9'(t) \leqslant -\left(\frac{lN_3}{2} - c\right)\int_0^L \psi_x^2dx - \left(\frac{\rho_1}{16} - \epsilon N_3\right)\int_0^L \varphi_t^2dx - \left(N_2\rho_2g_0 - \frac{cN_3}{\epsilon} - c\right)\int_0^L \psi_t^2dx - \frac{k_1}{8}\int_0^L(\varphi_x+\psi)^2dx$$
$$+ C_{N_2N_3}g_2(t)g^p\circ\psi_x + \left(\frac{N_1}{2} - cN_2 - c\right)g'\circ\psi_x + \frac{N_1}{2}g'\circ\psi_{xt} - N_1g(t)\int_0^L \psi_{0xx}\psi_{tt}dx + \left(\frac{\rho_1k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx.$$

At this point, we choose N_3 large enough so that $\frac{lN_3}{2} - c > 0$, then $\epsilon \in]0,1[$ small enough so that $\frac{\rho_1}{16} - \epsilon N_3 > 0$. Next, we pick N_2 so large that $N_2\rho_2g_0 - \frac{cN_3}{\epsilon} - c > 0$. Consequently, we have, for all $t \geqslant t_0$,

$$I_9'(t) \leqslant -c\int_0^L\left(\psi_x^2 + \varphi_t^2 + \psi_t^2 + (\varphi_x+\psi)^2\right)dx + cg_2(t)g^p\circ\psi_x + \left(\frac{N_1}{2} - c\right)g'\circ\psi_x + \frac{N_1}{2}g'\circ\psi_{xt} - N_1g(t)$$
$$\times\int_0^L \psi_{0xx}\psi_{tt}dx + \left(\frac{\rho_1k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t\psi_{xt}dx. \tag{3.19}$$

Now we estimate the last term of (3.19). We proceed as in [20] (in the case of past history control) and we prove the following lemma.

Lemma 3.12. *Let (φ,ψ) be the strong solution of(P) Then, for any $\varepsilon>0$ and $t\geqslant t_0$, we have*

$$\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx\leqslant \varepsilon\int_0^L \varphi_t^2dx+\frac{c}{\varepsilon}(g_2(t)g^p\circ\psi_{xt}-g'\circ\psi_x)+\frac{c}{\varepsilon}E(0)g(t). \tag{3.20}$$

Proof. We have, for all $t\geqslant t_0$,

$$\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_t\psi_{xt}dx=\frac{\frac{\rho_1k_2}{k_1}-\rho_2}{\int_0^t g(s)ds}\int_0^L\varphi_t\int_0^t g(t-s)(\psi_{xt}(t)-\psi_{xt}(s))dsdx+\frac{\frac{\rho_1k_2}{k_1}-\rho_2}{\int_0^t g(s)ds}\int_0^L\varphi_t\int_0^t g(t-s)\psi_{xt}(s)dsdx.$$

By noting that $\frac{1}{\int_0^t g(s)ds}\leqslant\frac{1}{g_0}$, for all $t\geqslant t_0$, exploiting Young's inequality and (3.3) (for ψ_{xt}), we get, for all $\varepsilon>0$ and $t\geqslant t_0$,

$$\frac{\frac{\rho_1k_2}{k_1}-\rho_2}{\int_0^t g(s)ds}\int_0^L\varphi_t\int_0^t g(t-s)(\psi_{xt}(t)-\psi_{xt}(s))dsdx\leqslant\frac{\varepsilon}{2}\int_0^L\varphi_t^2dx+\frac{c}{\varepsilon}g_2(t)g^p\circ\psi_{xt}.$$

On the other hand, by integrating by parts and using (3.4) and the fact that E and g are non-increasing and $\psi(x,0)=\psi_0(x)$, we obtain

$$\begin{aligned}\frac{\frac{\rho_1k_2}{k_1}-\rho_2}{\int_0^t g(s)ds}\int_0^L\varphi_t\int_0^t g(t-s)\psi_{xt}(s)dsdx&=\frac{\frac{\rho_1k_2}{k_1}-\rho_2}{\int_0^t g(s)ds}\int_0^L\varphi_t\left(g(0)\psi_x-g(t)\psi_{0x}+\int_0^t g'(t-s)\psi_x(s)ds\right)dx\\&=\frac{\frac{\rho_1k_2}{k_1}-\rho_2}{\int_0^t g(s)ds}\int_0^L\varphi_t\left(g(t)(\psi_x-\psi_{0x})-\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx\\&\leqslant\frac{\varepsilon}{2}\int_0^L\varphi_t^2dx+\frac{c}{\varepsilon}g(t)\int_0^L(\psi_{0x}^2+\psi_x^2)dx-\frac{c}{\varepsilon}g'\circ\psi_x\\&\leqslant\frac{\varepsilon}{2}\int_0^L\varphi_t^2dx+\frac{c}{\varepsilon}g(t)E(0)-\frac{c}{\varepsilon}g'\circ\psi_x.\end{aligned}$$

Inserting these last two inequalities into the first equality, we obtain (3.20). □

Lemma 3.13. *Let (φ,ψ) be the strong solution of (P). Then, for any $t\geqslant t_0$, we have*

$$I_9'(t)\leqslant -c\int_0^L\left(\psi_x^2+\varphi_t^2+\psi_t^2+(\varphi_x+\psi)^2\right)dx+cg_2(t)(g^p\circ\psi_x+g^p\circ\psi_{xt})+c\left[E(0)+E_*(0)+\int_0^L\psi_{0xx}^2dx\right]g(t). \tag{3.21}$$

Proof. We have, using Young's inequality and (3.18),

$$-\int_0^L\psi_{0xx}\psi_{tt}dx\leqslant\frac{1}{2}\int_0^L(\psi_{tt}^2+\psi_{0xx}^2)dx\leqslant c\left(E_*(t)+\int_0^L\psi_{0xx}^2dx\right)\leqslant c\left(E_*(0)+\int_0^L\psi_{0xx}^2dx\right). \tag{3.22}$$

Then, inserting (3.22) and (3.20) into (3.19) and choosing ε small enough such that $\varepsilon<c$, and N_1 large enough such that $I_9\geqslant cE$ and $\frac{N_1}{2}-c-\frac{c}{\varepsilon}>0$, we obtain (3.21). □

Now, using (3.2), we have

$$\begin{aligned}E^{2p-1}(t)&\leqslant c\left(\int_0^L\left(\psi_x^2+\varphi_t^2+\psi_t^2+(\varphi_x+\psi)^2\right)dx\right)^{2p-1}+c(g\circ\psi_x)^{2p-1}\\&\leqslant cE^{2(p-1)}(0)\left(\int_0^L\left(\psi_x^2+\varphi_t^2+\psi_t^2+(\varphi_x+\psi)^2\right)dx+g_1(t)g^p\circ\psi_x\right),\end{aligned}$$

hence

$$-E^{2(p-1)}(0)\left(\int_0^L\left(\psi_x^2+\varphi_t^2+\psi_t^2+(\varphi_x+\psi)^2\right)dx\right)\leqslant -cE^{2p-1}(t)+E^{2(p-1)}(0)g_1(t)g^p\circ\psi_x. \tag{3.23}$$

Multiplying (3.21) by $E^{2(p-1)}(0)$ and inserting (3.23), we get, for all $t\geqslant t_0$,

$$E^{2(p-1)}(0)I_9'(t) \leqslant -cE^{2p-1}(t) + cE^{2(p-1)}(0)[(g_1(t)+g_2(t))g^p \circ \psi_x + g_2(t)g^p \circ \psi_{xt}] + cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)g(t). \quad (3.24)$$

We note that by condition (2.2) and the fact that ξ is non-increasing, we have

$$\xi(g^p \circ v) \leqslant (\xi g^p)\circ v \leqslant -g' \circ v, \qquad v \in \{\psi_x, \psi_{xt}\}.$$

Then we multiply (3.24) by ξ to get, for all $t \geqslant t_0$,

$$\xi(t)E^{2p-1}(t) \leqslant -cE^{2(p-1)}(0)\xi(t)I_9'(t) - cE^{2(p-1)}(0)((g_1(t)+g_2(t))g' \circ \psi_x + g_2(t)g' \circ \psi_{xt}) + cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\xi(t)g(t).$$

By integrating over $[t_0, t]$, using (3.1), (3.17), (3.18), (3.22) and the fact that E^{2p-1} and ξ are non-increasing, $0 \leqslant I_9 \leqslant c(E+E_*)$ and $\int_0^{+\infty} g(s)ds < +\infty$, we obtain, for all $t \geqslant t_0$,

$$\begin{aligned}
E^{2p-1}(t)\int_0^t \xi(s)ds &\leqslant \int_0^t \xi(s)E^{2p-1}(s)ds = \int_0^{t_0} \xi(s)E^{2p-1}(s)ds + \int_{t_0}^t \xi(s)E^{2p-1}(s)ds \\
&\leqslant t_0\xi(0)E^{2p-1}(0) + cE^{2(p-1)}(0)\left(\xi(t_0)I_9(t_0) - \xi(t)I_9(t) + \int_{t_0}^t \xi'(s)I_9(s)ds\right) - cE^{2(p-1)}(0) \\
&\quad \times \int_{t_0}^t \left[(g_1(s)+g_2(s))E'(s) + g_2(s)\left(E_*'(s) + g(s)\int_0^L \psi_{0xx}\psi_{tt}dx\right)\right]ds \\
&\quad + cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\int_{t_0}^t \xi(s)g(s)ds \\
&\leqslant cE^{2(p-1)}(0)(E(0)+E(t_0)+E_*(t_0)) \\
&\quad + cE^{2(p-1)}(0)[(g_1(t_0)+g_2(t_0))E(t_0) - (g_1(t)+g_2(t))E(t) + g_2(t_0)E_*(t_0) - g_2(t)E_*(t)] + cE^{2(p-1)}(0) \\
&\quad \times \int_{t_0}^t [(g_1'(s)+g_2'(s))E(s) + g_2'(s)E_*(s)]ds + E^{2(p-1)}(0)\left(E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\int_{t_0}^t g_2(s)g(s)ds \\
&\quad + cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\int_0^t g(s)ds \\
&\leqslant cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\left(1+\int_{t_0}^t (g_1'(s)+g_2'(s)+g_2(s)g(s))ds\right) \\
&\leqslant cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)\left(1+g_1(t)+g_2(t)+\int_0^t g_2(s)g(s)ds\right).
\end{aligned}$$

Since g^{1-p} is non-decreasing then

$$\int_0^t g_2(s)g(s)ds = \int_0^t g(s)\left(\int_0^s g^{2-p}(\tau)d\tau\right)ds \leqslant \int_0^t g^{2-p}(s)\left(\int_0^s g(\tau)d\tau\right)ds \leqslant \left(\int_0^{+\infty} g(s)ds\right)g_2(t) \leqslant cg_2(t).$$

Consequently, we deduce, for all $t \geqslant t_0$,

$$E^{2p-1}(t)\int_0^t \xi(s)ds \leqslant cE^{2(p-1)}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)(1+g_1(t)+g_2(t)),$$

which gives (2.7) for all $t \geqslant t_0$ with

$$C = cE^{\frac{2(p-1)}{2p-1}}(0)\left(E(0)+E_*(0)+\int_0^L \psi_{0xx}^2 dx\right)^{\frac{1}{2p-1}}.$$

Thanks to the continuity and the boundedness of E, (2.7) holds, for all $t > 0$. This completes the proof of Theorem 2.1. □

Remark 3.1. Our stability result also holds for the following boundary conditions:

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0 \quad \text{on} \quad \mathbb{R}_+. \quad (3.25)$$

For this purpose, we introduce a new dependent variable. Namely,

$$\tilde{\psi}(x,t) = \psi(x,t) - \frac{1}{L}\left(\int_0^L \psi_0(x)dx\right)\cos\sqrt{\frac{k_1}{\rho_2}}t - \frac{1}{L}\sqrt{\frac{\rho_2}{k_1}}\left(\int_0^L \psi_1(x)dx\right)\sin\sqrt{\frac{k_1}{\rho_2}}t.$$

As a result, $(\varphi, \tilde{\psi})$ satisfies system (P), with $\tilde{\psi}_x(0,t) = \tilde{\psi}_x(L,t) = 0$, and more importantly,

$$\int_0^L \tilde{\psi}(x,t)dx = 0, \quad \forall \quad t \geqslant 0;$$

which allows the application of Poinaré's inequality. The proof, in this case, is simpler because we have

$$\left[\left(k_2\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x + \psi)\right]_0^L = 0,$$

and then (3.7) becomes

$$I_3'(t) \leqslant -k_1 \int_0^L (\varphi_x + \psi)^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \epsilon \int_0^L \varphi_t^2 dx + \frac{c}{\epsilon} \int_0^L \psi_x^2 dx - cg' \circ \psi_x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_t \psi_{xt} dx.$$

Therefore, we take $I_4 = I_5 = 0$ in Lemma 3.6, $I_6 = I_3$ in Lemma 3.7, and we complete the proof exactly as before. Note that we do not require that

$$\int_0^L \psi_0(x)dx = \int_0^L \psi_1(x)dx = 0.$$

Remark 3.2. Similarly, Our stability result also holds for the boundary conditions:

$$\varphi_x(0,t) = \varphi_x(L,t) = \psi(0,t) = \psi(L,t) = 0 \quad \text{on} \quad \mathbb{R}_+. \tag{3.26}$$

In this case, we introduce a new dependent variable. Namely,

$$\overline{\varphi}(x,t) = \varphi(x,t) - \frac{1}{L} t \int_0^L \varphi_1(x)dx - \frac{1}{L} \int_0^L \varphi_0(x)dx$$

and proceed exactly like the case of the boundary conditions (3.25). Again, we do not require that

$$\int_0^L \varphi_0(x)dx = \int_0^L \varphi_1(x)dx = 0.$$

Remark 3.3. Our result is still true if we consider (P), with Dirichlet homogeneous boundary conditions or (3.25) or (3.26) and with ρ_1, ρ_2, k_1, k_2 depending only on the space variable such that $\rho_1, \rho_2 \in C([0,L])$ and $k_1, k_2 \in C^1([0,L])$ satisfying

$$\text{Inf}_{x\in]0,L[}\ \rho_i(x) > 0, \quad \text{Inf}_{x\in]0,L[}\ k_i(x) > 0, \quad (i = 1,2),$$

$$\text{Inf}_{x\in]0,L[}\ k_2(x) - \int_0^{+\infty} g(t)dt > 0.$$

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