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# On the control of a viscoelastic damped Timoshenko-type system

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ABSTRACT

In this paper we consider the following Timoshenko system

$$\varphi_{tt} - (\varphi_x + \psi)_x = 0, \quad (0,1) \times (0,+\infty)$$
$$\psi_{tt} - \psi_{xx} + \int_0^t g(t-\tau)\psi_{xx}(\tau)\mathrm{d}\tau + \varphi_x + \psi = 0, \quad (0,1) \times (0,+\infty)$$

with Dirichlet boundary conditions where $g$ is a positive nonincreasing function. We establish an exponential and polynomial decay results with weaker conditions on $g$ than those required in [F. Ammar-Khodja, A. Benabdallah, J.E. Muñoz Rivera, R. Racke, Energy decay for Timoshenko systems of memory type, J. Differ. Equations, 194 (2003) 82–115].


## 1. Introduction

In [1], Timoshenko gave the following system of coupled hyperbolic equations:

$$\begin{aligned} &\rho u_{tt} = (K(u_x - \varphi))_x, \text{ in } (0,L) \times (0,+\infty) \\ &I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), \text{ in } (0,L) \times (0,+\infty), \end{aligned} \tag{1.1}$$

as a simple model describing the transverse vibration of a beam. Where $t$ denotes the time variable and $x$ is the space variable along the beam of length $L$, in its equilibrium configuration, $u$ is the transverse displacement of the beam and $\varphi$ is the rotation angle of the filament of the beam. The coefficients $\rho, I_\rho, E, I$ and $K$ are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

This system has been studied by many mathematicians and results concerning existence and asymptotic behavior have been established. Kim and Renardy [2] considered (1.1) together with two linear boundary conditions of the form

$$\begin{aligned} &K\varphi(L,t) - K\frac{\partial u}{\partial x}(L,t) = \alpha\frac{\partial u}{\partial t}(L,t) \quad \forall t \geqslant 0 \\ &EI\frac{\partial \varphi}{\partial x}(L,t) = -\beta\frac{\partial \varphi}{\partial t}(L,t) \quad \forall t \geqslant 0 \end{aligned} \tag{1.2}$$

and established an exponential decay result. They also provided numerical estimates to the eigenvalues of the operator associated with system (1.1). An analogous result was also established by Feng et al. [3], where the stabilization of vibrations in a Timoshenko system was studied. Raposo et al. [4] studied (1.1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings and proved that the energy decays exponentially. This result is similar to the one by Taylor et al.

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[5] but, as they mentioned, the originality in their work lies on the semigroup theory method, which was developed by Liu and Zheng [6]. Soufyane and Wehbe [7] showed that it is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback. They considered

$$\begin{aligned} &\rho u_{tt} = (K(u_x - \varphi))_x, \text{ in } (0,L)\times(0,+\infty) \\ &I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t, \text{ in } (0,L)\times(0,+\infty) \\ &u(0,t) = u(L,t) = \varphi(0,t) = \varphi(L,t) = 0 \quad \forall t > 0, \end{aligned} \tag{1.3}$$

where $b$ is a positive and continuous function, which satisfies

$$b(x) \geqslant b_0 > 0 \quad \forall x \in [a_0, a_1] \subset [0, L].$$

In fact, they proved that the uniform stability of (1.3) holds if and only if the wave speeds are equal ($\frac{K}{\rho} = \frac{EI}{I_\rho}$); otherwise only the asymptotic stability has been proved. This result improves an earlier one by Soufyane [8] and Shi and Feng [9], where an exponential decay of the solution energy of (1.1) together, with two locally distributed feedbacks, had been proved.

Ammar-Khodja et al. [10] considered a linear Timoshenko-type system with memory of the form

$$\begin{aligned} &\rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0 \\ &\rho_2 \psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)\mathrm{d}s + K(\varphi_x + \psi) = 0 \end{aligned} \tag{1.4}$$

in $(0,L)\times(0,+\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that, for a uniformly decaying relation function, the system is uniformly stable if and only if the wave speeds are equal; that is $\frac{K}{\rho_1} = \frac{b}{\rho_2}$. Precisely, they proved an exponential decay if $g$ satisfies a relation of the form

$$-k_0 g \leqslant g' \leqslant -k_1 g, \quad |g''| \leqslant k_2 g$$

for $k_0, k_1, k_2 > 0$ and a polynomial decay result if $g$ satisfies a relation of the form

$$\begin{aligned} &-b_1 g^{(p+1)/p} \leqslant g' \leqslant -b_2 g^{(p+1)/p}, \quad p > 2 \\ &-b_3 |g'|^{(p+2)/(p+1)} \leqslant g'' \leqslant -b_4 |g'|^{(p+2)/(p+1)}, \quad p > 2 \end{aligned}$$

for $b_1, b_2, b_3, b_4 > 0$. The feedback of memory type has also been used by Santos [11]. He considered a Timoshenko system and showed that the presence of two feedback of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. Shi and Feng [12] investigated a nonuniform Timoshenko beam and showed that, under some locally distributed controls, the vibration of the beam decays exponentially. To achieve their goal, the authors used the frequency multiplier method.

In the present work we are concerned with

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x = 0, \quad (0,1)\times\mathbb{R}_+ \\ \psi_{tt} - \psi_{xx} + \varphi_x + \psi + \int_0^t g(t-\tau)\psi_{xx}(\tau)\mathrm{d}\tau = 0, \quad (0,1)\times\mathbb{R}_+ \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad t \geqslant 0 \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,1) \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), \quad x \in (0,1). \end{cases} \tag{1.5}$$

Our aim in this work is to establish the same stabilization result of [10] with *weaker* conditions on $g$ (see Remark 3.1 by the end). Though we use the same method and adopt almost all the multipliers used in [10], the use of a functional similar to the one in [13,14] made the difference and played an essential role in weakening the requirements on $g$. We should note here that we do not loose generality by taking $\rho_1, \rho_2, K, b$, appeared in (1.4), to be equal to one and our argument also works for $K/\rho_1 = b/\rho_2$. The paper is organized as follows. In Section 2, We present some notations and material needed for our work and state our main result. The proof will be given in Section 3.

## 2. Preliminaries

In order to state our main result we make the following hypotheses:

**H 1.** $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0.$$

**H 2.** There exist constants $\xi > 0$ and $1 \leqslant p < 3/2$ such that

$$g'(s) \leqslant -\xi g^p(s), \quad s \geqslant 0.$$

**Remark 2.1.** Hypothesis (H2) implies that

$$\int_0^{+\infty} g^{2-p}(s)\mathrm{d}s < +\infty.$$

For completeness we state, without proof, an existence and regularity result.

**Proposition 2.1.** *Let $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in H_0^1(0,1)\times L^2(0,1)$ be given. Assume that (H1) is satisfied, then problem (1.5) has a unique global (weak) solution*

$$\varphi,\psi \in C(\mathbb{R}_+;H_0^1(0,1))\cap C^1(\mathbb{R}_+;L^2(0,1)). \tag{2.1}$$

*Moreover, if*

$$(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in (H^2(0,1)\cap H_0^1(0,1))\times H_0^1(0,1)$$

*then the solution satisfies*

$$\varphi,\psi \in C(\mathbb{R}_+;H^2(0,1)\cap H_0^1(0,1))\cap C^1(\mathbb{R}_+;H_0^1(0,1))\cap C^2(\mathbb{R}_+;L^2(0,1)).$$

**Remark 2.2.** This result can be proved using the Galerkin method.

Now, we introduce the energy functional

$$E(t) := \frac{1}{2}\int_0^1 \left[\varphi_t^2+\psi_t^2+\left[1-\int_0^t g(s)\mathrm{d}s\right]\psi_x^2+(\varphi_x+\psi)^2\right]\mathrm{d}x+\frac{1}{2}(g\circ\psi_x), \tag{2.2}$$

where for all $v\in L^2(0,1)$ and for all $1\leqslant p<3/2$,

$$(g^p\circ v)(t) = \int_0^1\int_0^t g^p(t-s)(v(t)-v(s))^2\mathrm{d}s\,\mathrm{d}x. \tag{2.3}$$

We are now ready to state our main stability result.

**Theorem 2.2.** *Let $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in H_0^1(0,1)\times L^2(0,1)$ be given. Assume that (H1)* and *(H2) are satisfied, then there exist two positive constants $c$ and $\omega$, for which the solution of problem (1.5) satisfies*

$$E(t)\leqslant c e^{-\omega t}\quad \forall t\geqslant 0 \quad \text{if}\quad p=1, \tag{2.4}$$

*and*

$$E(t)\leqslant c(1+t)^{-\frac{1}{p-1}}\quad \forall t\geqslant 0\quad \text{if}\quad p\neq 1. \tag{2.5}$$

## 3. Proof of the main result

In this section we prove our main result. For this purpose we will establish several lemmas.

**Lemma 3.1.** *Let $(\varphi,\psi)$ be the solution of (1.5). Then the energy functional satisfies*

$$E'(t) = -\frac{1}{2}g(t)\int_0^1\psi_x^2\,\mathrm{d}x+\frac{1}{2}(g'\circ\psi_x)\leqslant 0. \tag{3.1}$$

**Proof.** By multiplying equations in (1.5) by $\varphi_t$ and $\psi_t$ respectively and integrating over $(0,1)$, using integration by parts, hypotheses (H1) and (H2) and some manipulations as in [14], we obtain (3.1) for any regular solution. This equality remains valid for weak solutions by simple density argument.

The key point to show the exponential and the polynomial decay is to construct a Lyapunov functional $\mathscr{L}$ equivalent to $E$ and satisfying, for positive constants $\lambda_1$ and $\lambda_2$,

$$\mathscr{L}'(t)\leqslant -\lambda_2\mathscr{L}^{\lambda_1}(t)\quad\forall t\geqslant 0.$$

For this, we define several functionals which allow us to obtain the needed estimates. To simplify the computations we set

$$g\odot v = \int_0^1\int_0^t g(t-s)(v(t)-v(s))\mathrm{d}s\,\mathrm{d}x$$

for all $v\in L^2(0,1)$ and use $c$, throughout this paper, to denote a generic positive constant. □

**Lemma 3.2.** *There exists a positive constant c such that*

$$(g \odot v)^2 \leqslant c g^p \circ v_x$$

*for all* $v \in H_0^1(0,1)$.

**Proof.** By using Hölder's inequality and Poincaré's inequality, we get

$$\begin{aligned}(g \odot v)^2 &= \left(\int_\Omega \int_0^t g^{1-\frac{p}{2}}(t-\tau) g^{\frac{p}{2}}(t-s)(v(t)-v(s))\,ds\,dx\right)^2 \\ &\leqslant c\left(\int_0^t g^{2-p}(s)\,ds\right)\left(\int_\Omega \int_0^t g^p(t-s)(v(t)-v(s))^2\,ds\,dx\right) \leqslant c g^p \circ v_x. \qquad \square\end{aligned}$$

**Lemma 3.3.** *Under the assumptions* (H1) and (H2), *the functional I defined by*

$$I(t) := -\int_0^1 \psi_t \int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\,dx$$

*satisfies, along the solution, the estimate*

$$I'(t) \leqslant -\left(\int_0^t g(s)\,ds - \delta\right)\int_0^1 \psi_t^2\,dx + \delta\int_0^1 (\varphi_x+\psi)^2\,dx + c\delta\int_0^1 \psi_x^2\,dx - \frac{c}{\delta} g' \circ \psi_x + c\left(\delta + \frac{1}{\delta}\right) g^p \circ \psi_x, \tag{3.2}$$

*for all* $\delta > 0$.

**Proof.** By using equations in (1.5), we get

$$\begin{aligned}I'(t) &= -\int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,ds\,dx - \left(\int_0^t g(s)\,ds\right)\int_0^1 \psi_t^2\,dx \\ &\quad - \int_0^1 \left[\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)\,ds - \varphi_x - \psi\right]\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\,dx \\ &= -\int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,ds\,dx - \left(\int_0^t g(s)\,ds\right)\int_0^1 \psi_t^2\,dx \\ &\quad + \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\,dx + \int_0^1 (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\,dx \\ &\quad - \int_0^1 \left(\int_0^t g(t-s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx.\end{aligned}$$

We now estimate the terms in the right side of the above equality as follows.

By using Young's inequality and Lemma 3.2 (for $g'$ and $p = 1$) we obtain, for all $\delta > 0$,

$$-\int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,ds\,dx \leqslant \delta\int_0^1 \psi_t^2\,dx - \frac{c}{\delta} g' \circ \psi_x.$$

Similarly, we have

$$\begin{aligned}&-\int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\,dx \leqslant \delta\int_0^1 \psi_x^2\,dx + \frac{c}{\delta} g^p \circ \psi_x, \\ &-\int_0^1 (\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,ds\,dx \leqslant \delta\int_0^1 (\varphi_x+\psi)^2\,dx + \frac{c}{\delta} g^p \circ \psi_x,\end{aligned}$$

and

$$\begin{aligned}&-\int_0^1 \left(\int_0^t g(t-s)\psi_x(s)\,ds\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)dx \\ &\leqslant \delta'\int_0^1 \left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t)+\psi_x(t))\,ds\right)^2 dx + \frac{c}{\delta'}\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx \\ &\leqslant 2\delta'\int_0^1 \psi_x^2\left(\int_0^t g(s)\,ds\right)^2 dx + \left(2\delta' + \frac{c}{\delta'}\right)\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,ds\right)^2 dx \\ &\leqslant c\delta'\int_0^1 \psi_x^2\,dx + c\left(\delta' + \frac{1}{\delta'}\right) g^p \circ \psi_x \leqslant \delta\int_0^1 \psi_x^2\,dx + c\left(\delta + \frac{1}{\delta}\right) g^p \circ \psi_x,\end{aligned}$$

By combining all the above estimates, the assertion of Lemma 3.3 is proved. $\square$

**Lemma 3.4.** *Under the assumptions* (H1) and (H2), *the functional J defined by*

$$J(t) := -\int_0^1 (\psi\psi_t + \varphi\varphi_t)\mathrm{d}x$$

*satisfies, along the solution, the estimate*

$$J'(t) \leqslant -\int_0^1 (\psi_t^2 + \varphi_t^2)\mathrm{d}x + \int_0^1 (\psi + \varphi_x)^2\,\mathrm{d}x + c\int_0^1 \psi_x^2\,\mathrm{d}x + cg^p \circ \psi_x. \tag{3.3}$$

**Proof.** By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned} J'(t) &= -\int_0^1 (\psi_t^2 + \varphi_t^2)\mathrm{d}x - \int_0^1 \varphi(\psi_x + \varphi_{xx})\mathrm{d}x - \int_0^1 \psi\left[\psi_{xx} - \int_0^t g(t-s)(\psi_x(s))_x \mathrm{d}s - \varphi_x - \psi\right]\mathrm{d}x \\ &= -\int_0^1 (\psi_t^2 + \varphi_t^2)\mathrm{d}x + \int_0^1 \psi_x^2\,\mathrm{d}x - \int_0^1 \psi_x\left(\int_0^t g(t-s)\psi_x(s)\mathrm{d}s\right)\mathrm{d}x + \int_0^1 (\psi + \varphi_x)^2\,\mathrm{d}x \\ &\leqslant -\int_0^1 (\psi_t^2 + \varphi_t^2)\mathrm{d}x + \int_0^1 (\psi + \varphi_x)^2 + c\int_0^1 \psi_x^2\,\mathrm{d}x + cg^p \circ \psi_x. \end{aligned}$$

This completes the proof of Lemma 3.4. □

**Lemma 3.5.** *Assume that* (H1) and (H2) *hold. Then, the functional K defined by*

$$K(t) := \int_0^1 \psi_t(\psi + \varphi_x)\mathrm{d}x + \int_0^1 \psi_x\varphi_t\,\mathrm{d}x - \int_0^1 \varphi_t \int_0^t g(t-s)\psi_x(s)\mathrm{d}s\,\mathrm{d}x$$

*satisfies, along the solution, the estimate*

$$\begin{aligned} K'(t) \leqslant{} & \left[(\psi_x - \int_0^t g(t-s)\psi_x(s)\mathrm{d}s)\varphi_x\right]_{x=0}^{x=1} - (1-\varepsilon)\int_0^1 (\psi + \varphi_x)^2\,\mathrm{d}x + \varepsilon\int_0^1 \varphi_t^2\,\mathrm{d}x - \frac{c}{\varepsilon}g' \circ \psi_x \\ & + \frac{c}{\varepsilon}\int_0^1 \psi_x^2\,\mathrm{d}x + \int_0^1 \psi_t^2\,\mathrm{d}x \end{aligned} \tag{3.4}$$

*for any* $0 < \varepsilon < 1$.

**Proof.** By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned} K'(t) = {} & \int_0^1 (\varphi_x + \psi)\left[\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)\mathrm{d}s - \varphi_x - \psi\right]\mathrm{d}x \\ & + \int_0^1 (\varphi_{xt} + \psi_t)\psi_t\,\mathrm{d}x + \int_0^1 \psi_{xt}\varphi_t\,\mathrm{d}x + \int_0^1 \psi_x(\varphi_x + \psi)_x\,\mathrm{d}x - \int_0^1 (\varphi_x + \psi)_x \int_0^t g(t-s)\psi_x(s)\mathrm{d}s\,\mathrm{d}x \\ & - \int_0^1 \varphi_t(g(0)\psi_x + \int_0^t g'(t-s)\psi_x(s)\mathrm{d}s)\mathrm{d}x = \left[(\psi_x - \int_0^t g(t-s)\psi_x(s)\mathrm{d}s)\varphi_x\right]_{x=0}^{x=1} \\ & - \int_0^1 (\psi + \varphi_x)^2\,\mathrm{d}x + \int_0^1 \psi_t^2\,\mathrm{d}x + g(t)\int_0^1 \psi_x\varphi_t\,\mathrm{d}x - \int_0^1 \varphi_t \int_0^t g'(t-s)(\psi_x(s) - \psi_x(t))\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

By using Young's inequality, (3.4) is established. □

**Lemma 3.6.** *Assume that* (H1) and (H2) *hold. Let* $m \in C^1([0,1])$ *be a function satisfying* $m(0) = -m(1) = 2$. *Then there exists* $c > 0$ *such that for any* $\varepsilon > 0$ *we have, along the solution,*

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 m(x)\psi_t(\psi_x - \int_0^t g(t-s)\psi_x(s)\mathrm{d}s)\mathrm{d}x \\ & \leqslant -\left((\psi_x(1,t) - \int_0^t g(t-s)\psi_x(1,s)\mathrm{d}s)^2 + (\psi_x(0,t) - \int_0^t g(t-s)\psi_x(0,s)\mathrm{d}s)^2\right) \\ & \quad + \varepsilon\int_0^1 (\psi + \varphi_x)^2\,\mathrm{d}x + \frac{c}{\varepsilon}\left(\int_0^1 \psi_x^2\,\mathrm{d}x + g^p \circ \psi_x\right) + c\left(\int_0^1 \psi_t^2\,\mathrm{d}x - g' \circ \psi_x\right) \end{aligned}$$

*and*

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 m(x)\varphi_t\varphi_x\,\mathrm{d}x \leqslant -(\varphi_x^2(1,t) + \varphi_x^2(0,t)) \\ & + c\int_0^1 (\varphi_t^2 + \varphi_x^2 + \psi_x^2)\mathrm{d}x. \end{aligned}$$

**Proof.** By exploiting equations (1.5) and repeating the same procedure as in above, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 m(x)\psi_t\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)dx\\
&=\int_0^1 m(x)\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)_x\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)dx\\
&\quad-\int_0^1 m(x)\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x+\psi)dx\\
&\quad+\int_0^1 m(x)\psi_t\left(\psi_{xt}-a(x)g(0)\psi_x-\int_0^t g'(t-s)\psi_x(s)ds\right)dx\\
&=-\left(\left(\psi_x(1,t)-\int_0^t g(t-s)\psi_x(1,s)ds\right)^2+\left(\psi_x(0,t)-\int_0^t g(t-s)\psi_x(0,s)ds\right)^2\right)\\
&\quad-\frac{1}{2}\int_0^1 m'(x)\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2dx\\
&\quad-\int_0^1 m(x)\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)(\varphi_x+\psi)dx-\frac{1}{2}\int_0^1 m'(x)\psi_t^2\,dx\\
&\quad+\int_0^1 m(x)\psi_t\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))ds\right)dx+g(t)\int_0^1 m(x)\psi_x\psi_t\,dx.
\end{aligned}$$

By using Young's inequality and Lemma 3.2, the first estimate of Lemma 3.6 is established. □

Similarly, we can prove the second estimate of Lemma 3.7.

**Lemma 3.7.** *Assume that* (H1) and (H2) *hold. Then, the functional L defined by*

$$L(t):=K(t)+\frac{1}{4\varepsilon}\int_0^1 m(x)\psi_t\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)dx+\varepsilon\int_0^1 m(x)\varphi_t\varphi_x\,dx$$

*satisfies, along the solution, the estimate*

$$L'(t)\leqslant-\left(\frac{3}{4}-c\varepsilon\right)\int_0^1(\varphi_x+\psi)^2\,dx+c\varepsilon\int_0^1\varphi_t^2\,dx+\frac{c}{\varepsilon}\int_0^1\psi_t^2\,dx+\frac{c}{\varepsilon^2}\int_0^1\psi_x^2\,dx-\frac{c}{\varepsilon}g'\circ\psi_x+\frac{c}{\varepsilon^2}g^p\circ\psi_x \tag{3.5}$$

*for any* $0<\varepsilon<1$.

**Proof.** By using Lemmas 3.5 and 3.6, Young's and Poincaré's inequalities, and the fact that

$$\varphi_x^2\leqslant 2(\psi+\varphi_x)^2+2\psi^2$$

and

$$(\psi_x-\int_0^t g(t-s)\psi_x(s)ds)\varphi_x\leqslant\varepsilon\varphi_x^2+\frac{1}{4\varepsilon}\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2,$$

we obtain (3.5).

Let $L_1(t):=L(t)+2c\varepsilon J(t)$. By using Lemmas 3.4 and 3.7, and fixing $\varepsilon$ small enough, we obtain

$$L_1'(t)\leqslant-\frac{1}{2}\int_0^1(\psi+\varphi_x)^2dx-\tau\int_0^1\varphi_t^2\,dx+c\int_0^1\psi_t^2\,dx+c\int_0^1\psi_x^2\,dx+cg^p\circ\psi_x-cg'\circ\psi_x \tag{3.6}$$

where $\tau=c\varepsilon$.

As in [10], we use the multiplier $w$ given by the solution of

$$-w_{xx}=\psi_x,\quad w(0)=w(1)=0.\qquad\square \tag{3.7}$$

**Lemma 3.8.** *The solution of* (3.7) *satisfies*

$$\int_0^1 w_x^2\,dx\leqslant\int_0^1\psi^2\,dx$$

*and*

$$\int_0^1 w_t^2dx\leqslant\int_0^1\psi_t^2\,dx.$$

**Proof.** We multiply Eq. (3.7) by $w$, integrate by parts, and use the Cauchy-Schwarz inequality, to get

$$\int_0^1 w_x^2\,dx \leqslant \int_0^1 \psi^2\,dx.$$

Next, we differentiate (3.7) with respect to $t$ to obtain, by similar calculations,

$$\int_0^1 w_{xt}^2\,dx \leqslant \int_0^1 \psi_t^2\,dx.$$

Poincaré's inequality, then yields

$$\int_0^1 w_t^2\,dx \leqslant \int_0^1 \psi_t^2\,dx.$$

This completes the proof of Lemma 3.6. □

**Lemma 3.9.** *Under the assumptions* (H1) and (H2), *the functional* $J_1$ *defined by*

$$J_1(t) := \int_0^1 (\psi\psi_t + w\varphi_t)dx$$

*satisfies, along the solution, the estimate*

$$J_1'(t) \leqslant -\frac{l}{2}\int_0^1 \psi_x^2\,dx + \frac{c}{\varepsilon_1}\int_0^1 \psi_t^2\,dx + \varepsilon_1\int_0^1 \varphi_t^2\,dx + cg^p \circ \psi_x \tag{3.8}$$

*for any* $0 < \varepsilon_1 < l$ *(l is defined in* (H1)).

**Proof.** By exploiting (1.5) and integrating by parts, we have

$$\begin{aligned} J_1'(t) &= \int_0^1 (\psi_t^2 - \psi_x^2)dx + \int_0^1 \psi_x \int_0^t g(t-s)\psi_x(s)ds\,dx - \int_0^1 \psi(\psi+\varphi_x)dx - \int_0^1 w_x(\psi+\varphi_x)dx + \int_0^1 w_t\varphi_t\,dx \\ &\leqslant \int_0^1 \psi_t^2\,dx - \frac{l}{2}\int_0^1 \psi_x^2\,dx + cg^p\circ\psi_x + \int_0^1 (w_x^2 - \psi^2)dx + \frac{c}{\varepsilon_1}\int_0^1 \varphi_t^2\,dx + \varepsilon_1\int_0^1 w_t^2. \end{aligned}$$

Lemma 3.8 gives the desired result. □

For $N_1, N_2, N_3 > 1$, let

$$\mathscr{L}(t) := N_1E(t) + N_2I(t) + N_3J_1 + L_1(t)$$

and $g_0 = \int_0^{t_0} g(s)ds > 0$ for some fixed $t_0 > 0$. By combining (3.1), (3.2), (3.6), (3.8), and taking $\delta = \frac{1}{4N_2}$, we arrive at

$$\begin{aligned} \mathscr{L}'(t) \leqslant& -\left(N_2g_0 - \frac{1}{4}\right)\int_0^1 \psi_t^2\,dx + c\frac{N_3}{\varepsilon_1}\int_0^1 \psi_t^2\,dx - \left(\frac{N_3}{2} - c - \frac{c}{N_2}\right)\int_0^1 \psi_x^2\,dx - (c - N_3\varepsilon_1)\int_0^1 \varphi_t^2 dx \\ &- \frac{1}{4}\int_0^1 (\psi+\varphi_x)^2\,dx + \left(\frac{N_1}{2} - cN_2^2\right)g'\circ\psi_x + c(N_2^2+N_3)g^p\circ\psi_x \end{aligned} \tag{3.9}$$

for all $t \geqslant t_0$.

We distinguish two cases:

**Case 1.** $p = 1$. In this case, we choose $N_3$ large enough so that

$$\frac{N_3}{2} > c,$$

then $\varepsilon_1$ small enough so that

$$\varepsilon_1 < \frac{c}{N_3}.$$

Next, we choose $N_2$ large enough so that

$$N_2g_0 - \frac{1}{4} > \frac{2cN_3}{\varepsilon_1}, \quad \frac{N_3}{2} - c - \frac{c}{N_2} > 0.$$

Finally, we choose $N_1$ large enough so that

$$N_1c_1 - c(N_2^2 + N_3) > N_2g_0 - \frac{1}{4}, \quad \xi\left(\frac{N_1}{2} - cN_2^2\right) > c(N_2^2 + N_3).$$

Therefore (3.9) takes the form

$$\mathscr{L}'(t) \leqslant -\left(N_2 g_0 - \frac{1}{4} - c\frac{N_3}{\varepsilon_1}\right)\int_0^1 \psi_t^2\,dx - \left(\frac{N_3}{2} - c - \frac{c}{N_2}\right)\int_0^1 \psi_x^2\,dx - (c - N_3\varepsilon_1)\int_0^1 \varphi_t^2\,dx - \frac{1}{4}\int_0^1 (\psi+\varphi_x)^2\,dx - cg\circ\psi_x. \leqslant -cE(t),$$

for all $t \geqslant t_0$.

In the other hand, we can choose $N_1$ even larger (if needed) so that

$$\mathscr{L}(t) \sim E(t). \tag{3.10}$$

Therefore, by combining the last two inequalities, we obtain, for a positive constant $\omega$,

$$\mathscr{L}'(t) \leqslant -\omega\mathscr{L}(t), \quad t \geqslant t_0.$$

A simple integration over $(t_0, t)$, leads to

$$\mathscr{L}(t) \leqslant ce^{-\omega t}, \quad t \geqslant t_0.$$

Consequently, (2.4) is established by virtue of (3.10) and the continuity of $E$ over $[0, t_0]$.

**Case 2.** $p > 1$. With the same choice of constants as in Case 1, we deduce, from (3.9),

$$\mathscr{L}'(t) \leqslant -c\left(\int_0^1 \psi_t^2\,dx + \int_0^1 \psi_x^2\,dx + \int_0^1 \varphi_t^2\,dx + \int_0^1 (\psi+\varphi_x)^2\,dx + g^p\circ\psi_x\right). \tag{3.11}$$

But using (H1) and (H2), we easily see that

$$\int_0^\infty g^{1-\theta}(s)ds < \infty, \quad \theta < 2-p,$$

so Lemma 3.3 [15] yields

$$g\circ\psi_x \leqslant c\left\{\left(\int_0^\infty g^{1-\theta}(s)ds\right)E(0)\right\}^{(p-1)/(p-1+\theta)}\{g^p\circ\psi_x\}^{\theta/(p-1+\theta)}.$$

Therefore we get, for $\gamma \geqslant 1$,

$$\begin{aligned} E^\gamma(t) &\leqslant cE^{\gamma-1}(0)\left(\int_0^1 \psi_t^2\,dx + \int_0^1 \psi_x^2\,dx + \int_0^1 \varphi_t^2\,dx + \int_0^1 (\psi+\varphi_x)^2\,dx\right) + (g\circ\psi_x)^\gamma \\ &\leqslant cE^{\gamma-1}(0)\left(\int_0^1 \psi_t^2\,dx + \int_0^1 \psi_x^2\,dx + \int_0^1 \varphi_t^2\,dx + \int_0^1 (\psi+\varphi_x)^2\,dx\right) \\ &\quad + c\left\{\left(\int_0^\infty g^{1-\theta}(s)ds\right)E(0)\right\}^{\gamma(p-1)/(p-1+\theta)}\{g^p\circ\psi_x\}^{\theta\gamma/(p-1+\theta)} \end{aligned} \tag{3.12}$$

By choosing $\theta = \frac{1}{2}$ and $\gamma = 2p-1$ (hence $\gamma\theta/(p-1+\theta) = 1$), estimate (3.12) gives

$$E^\gamma(t) \leqslant c\left(\int_0^1 \psi_t^2\,dx + \int_0^1 \psi_x^2\,dx + \int_0^1 \varphi_t^2\,dx + \int_0^1 (\psi+\varphi_x)^2\,dx + g^p\circ\psi_x\right). \tag{3.13}$$

By combining (3.9), (3.10) and (3.12), we arrive at

$$\mathscr{L}'(t) \leqslant -c\mathscr{L}^\gamma(t), \quad t \geqslant t_0.$$

By integration, we get

$$\mathscr{L}(t) \leqslant -c(1+t)^{-\frac{1}{\gamma-1}}(t), \quad t \geqslant t_0. \tag{3.14}$$

As a consequence of (3.14), we have

$$\int_0^\infty \mathscr{L}(t)dt + \sup_{t\geqslant 0} t\mathscr{L}(t) < +\infty.$$

Therefore, by using again Lemma 3.3 of [15], we have

$$g\circ\psi_x \leqslant c\left(\int_0^t \|\psi(s)\|_{H^1(0,1)}ds + t\|\psi(t)\|_{H^1(0,1)}\right)^{\frac{p-1}{p}}(g^p\circ\psi_x)^{\frac{1}{p}} \leqslant c\left(\int_0^t \mathscr{L}(s)dt + t\mathscr{L}(t)\right)^{\frac{p-1}{p}}(g^p\circ\psi_x)^{\frac{1}{p}} \leqslant c(g^p\circ\psi_x)^{\frac{1}{p}},$$

which implies that

$$g^p\circ\psi_x \geqslant (g\circ\psi_x)^p.$$

So

$$\mathscr{L}'(t) \leqslant -c\left(\int_0^1 \psi_t^2\,dx + \int_0^1 \psi_x^2\,dx + \int_0^1 \varphi_t^2\,dx + \int_0^1 (\psi + \varphi_x)^2\,dx + (g \circ \psi_x)^p\right)$$

and, for (3.12) with $\gamma = p$,

$$E^p(t) \leqslant C\left(\int_0^1 \psi_t^2\,dx + \int_0^1 \psi_x^2 dx + \int_0^1 \varphi_t^2\,dx + \int_0^1 (\psi + \varphi_x)^2\,dx + (g \circ \psi_x)^p\right).$$

Combining the last two inequalities and (3.10), we obtain

$$\mathscr{L}'(t) \leqslant -c\mathscr{L}^p(t), \quad t \geqslant t_0.$$

A simple integration over $(t_0, t)$ and by virtue of boundedness of $\mathscr{L}$, we arrive at

$$\mathscr{L}(t) \leqslant c(1+t)^{-\frac{1}{p-1}}, \quad t \geqslant t_0.$$

Consequently, (2.5) is established by virtue of (3.10) and the continuity of $E$ over $[0, t_0]$.

**Remark 3.1.** We should note our result is established under weaker conditions on $g$ than those in [10]. Precisely, we do not require anything on $g''$ as in (1.6) and (1.7) of [10]. We only need $g$ to be differentiable satisfying (H1) and (H2).

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