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Corresponding Author	Family Name	**Guesmia**
	Particle	
	Given Name	**Aissa**
	Suffix	
	Division	Institut Elie Cartan de Lorraine (IECL), UMR 7502
	Organization	Université de Lorraine
	Address	Bat. A, Ile du Saulcy, 57045, Metz Cedex 01, France
	Phone	
	Fax	
	Email	aissa.guesmia@univ-lorraine.fr
	URL	
	ORCID	

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Abstract	In this paper, we consider a vibrating system of Timoshenko-type in a bounded one-dimensional domain with discrete time delay and complementary frictional damping and infinite memory controls all acting on the transversal displacement. We show that the system is well-posed in the sens of semigroup and that, under appropriate assumptions on the weights of the delay and the history data, the stability of the system holds in case of the equal-speed propagation as well as in the opposite case in spite of the presence of a discrete time delay, where the decay rate of solutions is given in terms of the smoothness of the initial data and the growth of the relaxation kernel at infinity. The results of this paper extend the ones obtained by the present author and Messaoudi in (Acta Math Sci 36:1–33, 2016) to the case of presence of discrete delay.

Keywords (separated by '-')	Well-posedness - General decay - Time delay - Infinite memory - Frictional damping - Viscoelastic - Timoshenko-type - Semigroup theory - Energy method
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Well-posedness and energy decay for Timoshenko systems with discrete time delay under frictional damping and/or infinite memory in the displacement

Aissa Guesmia[1]

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Abstract In this paper, we consider a vibrating system of Timoshenko-type in a bounded one-dimensional domain with discrete time delay and complementary frictional damping and infinite memory controls all acting on the transversal displacement. We show that the system is well-posed in the sens of semigroup and that, under appropriate assumptions on the weights of the delay and the history data, the stability of the system holds in case of the equal-speed propagation as well as in the opposite case in spite of the presence of a discrete time delay, where the decay rate of solutions is given in terms of the smoothness of the initial data and the growth of the relaxation kernel at infinity. The results of this paper extend the ones obtained by the present author and Messaoudi in (Acta Math Sci 36:1–33, 2016) to the case of presence of discrete delay.

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Mathematics Subject Classification 35B37 · 35L55 · 74D05 · 93D15 · 93D20

1 Introduction

In this paper, we are concerned with the well-posedness and the long-time behavior of the solution of the following Timoshenko system:

✉ Aissa Guesmia
aissa.guesmia@univ-lorraine.fr

1 Institut Elie Cartan de Lorraine (IECL), UMR 7502, Université de Lorraine, Bat. A, Ile du Saulcy, 57045 Metz Cedex 01, France

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - k_1(\varphi_x(x,t) + \psi(x,t))_x + d(x)\varphi_t(x,t-\tau) + b(x)\varphi_t(x,t) \\ \quad + \displaystyle\int_0^{+\infty} g(s)(a(x)\varphi_x(x,t-s))_x \, ds = 0, \\ \rho_2 \psi_{tt}(x,t) - k_2 \psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) = 0, \\ \varphi(0,t) = \psi_x(0,t) = \varphi(L,t) = \psi_x(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t), \ \varphi_t(x,0) = \varphi_1(x), \ \varphi_t(x,-\tau p) = f_0(x,-\tau p), \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \end{cases} \tag{1.1}$$

for $(x,t,p) \in]0,L[\times]0,+\infty[\times]0,1[$, $d : [0,L] \to \mathbb{R}$, $a, b : [0,L] \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions (to be specified later), where $\mathbb{R}_+ = [0,+\infty[$, L, τ, ρ_i, k_i $(i = 1,2)$ are positive constants,

$$\varphi_0 :]0,L[\times]-\infty,0[\to \mathbb{R}, \quad \varphi_1, \ \psi_0, \ \psi_1 :]0,L[\to \mathbb{R} \quad \text{and} \quad f_0 :]0,L[\times]-\tau,0[\to \mathbb{R}$$

are given initial data, and

$$(\varphi,\psi) :]0,L[\times]0,+\infty[\to \mathbb{R}^2$$

is the state of (1.1). A subscript y and the notation ∂_y denote the derivative with respect to y. We also use the prime notation to denote the derivative when the function has only one variable. The infinite integral in (1.1), $b(x)\varphi_t(x,t)$ and $d(x)\varphi_t(x,t-\tau)$ represent, respectively, the infinite memory, the frictional damping and the discrete time delay. For simplicity of notation, the space and time variables are used only when it is necessary to avoid ambiguity.

Our aim is the study of the well-posedness and asymptotic behavior of the solutions of (1.1) in case of the equal-speed propagation

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \tag{1.2}$$

as well as in the opposite case. The equality (1.2) means that the first two equations in (1.1) have the same speeds of wave propagation $\sqrt{\frac{k_1}{\rho_1}}$ and $\sqrt{\frac{k_2}{\rho_2}}$, respectively.

Timoshenko [69], in 1921, introduced the following model to describe the transverse vibration of a beam:

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in }]0,L[\times]0,+\infty[, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), & \text{in }]0,L[\times]0,+\infty[, \end{cases} \tag{1.3}$$

where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. Since then, this model has attracted the attention of many researchers and an important amount of work has been devoted to the issue of the stabilization and the search for the minimum dissipation by which the solutions decay uniformly to the stable state as time goes to infinity. To achieve this goal, diverse types of dissipative mechanisms have been used and several stability results have been obtained. We mention some of these results (for more results, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

Absence of delay: $d \equiv 0$. In the case of presence of controls on both the rotation angle and the transverse displacement, investigations showed that the Timoshenko systems are stable without any restriction on the constants ρ_1, ρ_2, k_1 and k_2. In this regards, many decay

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

estimates were obtained; see [26,31,39,40,56]. However, in the case of only one control on the rotation angle, the rate of decay depends heavily on the constants ρ_1, ρ_2, k_1 and k_2 and the regularity of the initial data. Precisely, if (1.2) holds, the results obtained are similar to those established for the case of the presence of controls in both equations. We quote in this regard [2,7,14,21–24,26,41,42,45–47,63]. But, if (1.2) does not hold, a situation which is more interesting from the physics point of view, then it has been shown that the Timoshenko system is not exponentially stable even for exponentially decaying relaxation functions or linear frictional damping, and only weak decay estimates can be obtained for regular solutions in the presence of dissipation. This has been demonstrated in [2,14,23,24,26,43], for the case of finite or infinite memory, and in [17,22], for complementary frictional damping and finite or infinite memory acting on the rotation angle equation. We also refer the reader to [55] (and its references) concerning the stability of Timoshenko-type systems in $\mathbb{R}$ (instaed of $]0, L[$) with controls acting on the rotation angle.

For the stability of Timoshenko systems via heat effect, we mention the pioneer work [44] devoted to the study of the following system:

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in }]0, L[\times]0, +\infty[, \end{cases} \tag{1.4}$$

where θ denotes the temperature difference. In their work, Rivera and Racke [44] established, under appropriate conditions on the function σ and the positive constants ρ_i, b, k and γ, several exponential decay results for the linearized system with various boundary conditions. They also proved a non-exponential stability result for the case of non-equal speed of propagation, and proved an exponential decay result for the nonlinear case. Guesmia et al. [27] discussed a linear version of (1.4) and completed the work of [44] by establishing some polynomial decay results in the case of non-equal speed of propagation.

In (1.4), the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation; that is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound. To overcome this physical paradox, many theories have merged. One of which suggests that we should replace Fourier's law by Cattaneo's law. In line with this theory, (1.4), in its linear form, becomes

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + \delta\theta_x = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \tau q_t + q + k\theta_x = 0, & \text{in }]0, L[\times]0, +\infty[, \end{cases} \tag{1.5}$$

where q denotes the heat flux. Fernández Sare and Racke [15] studied (1.5) and proved that (1.2) is no longer sufficient to obtain exponential stability even in the presence of an extra viscoelastic dissipation of the form $\int_0^{+\infty} g(s)\psi_{xx}(t-s)\,ds$ in the second equation. Very recently, Santos et al. [62] considered (1.5), introduced a new stability number

$$\chi = \left(\tau - \frac{\rho_1}{k_1\rho_3}\right)\left(\rho_2 - \frac{k_2\rho_1}{k_1}\right) - \frac{\tau\rho_1\delta^2}{k_1\rho_3} \tag{1.6}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

and used the semigroup method to obtain an exponential decay result, for $\chi = 0$, and a polynomial decay, for $\chi \neq 0$. See, also [26,29,30,39,54,58,59]. Notice that, when $\tau = 0$ (Fourier's law), $\chi = 0$ if and only if (1.2) holds.

In all above mentioned works, the stability was either via both equation control or the angular rotation equation control. Recently, Almeida Júnior et al. [4] considered the situation when the control is only on the transverse displacement equation, which is more realistic from the physics point of view. Precisely, they looked into the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + \mu\varphi_t = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) = 0, & \text{in }]0, L[\times]0, +\infty[, \end{cases} \tag{1.7}$$

where μ is a positive constant, and showed that the linear frictional damping $\mu\varphi_t$ is strong enough to obtain exponential stability of (1.7) provided that (1.2) holds. They, also, proved some non-exponential and polynomial decay results in the case of non-equal speed situation. The results of [4] were, very recently, extended in [25] to the case where the linear frictional damping $\mu\varphi_t$ is replaced by a nonlinear one and/or an infinite memory. The same authors of [4] considered in [5]

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \sigma\theta_x = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) - \sigma\theta = 0, & \text{in }]0, L[\times]0, +\infty[, \\ \rho_3\theta_t - \gamma\theta_{xx} + \sigma(\varphi_x + \psi)_t = 0, & \text{in }]0, L[\times]0, +\infty[, \end{cases} \tag{1.8}$$

with various boundary conditions, and established the exponential stability of (1.8) for equal-speed case, and non-exponential stability for the opposite case. In the case of lack of exponential stability, they proved some algebraic (polynomial) stability for strong solutions.

Presence of delay: $d \neq 0$. The questions related to well-posedness and stability/instability of Timoshenko-type systems as well as evolution equations with time delay have attracted considerable attention in recent years and many researchers have shown that the time delay can destabilize a system that was asymptotically stable in the absence of time delay.

When the delay and controls are present on the rotation angle equation, we mention the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + \int_0^t g(s)\psi_{xx}(t-s)\,ds + \mu_1\psi_t + \mu_2\psi_t(t-\tau) = 0, \end{cases} \tag{1.9}$$

in $]0, 1[\times]0, +\infty[$, studied in [57], where μ_1, μ_2 and τ are fixed non-negative constants. The author of [57] proved the stability of (1.9) under the assumptions (1.2) and $0 < \mu_2 \leq \mu_1$, where the decay rate of solutions depends on the one of g. The obtained stability results in [57] generalize the ones of [60] concerning (1.9) in the case $g \equiv 0$ and $0 < \mu_2 < \mu_1$, and they were generalized in [32] to the case $g \equiv 0$ and variable time delay $\tau(t)$. In [61], the stability of Timoshenko systems with two internal time delays and two boundary linear feedbacks was proved under some smallness conditions on L and the weights of the delays.

When no frictional damping is present, the stability of this Timoshenko system

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + \int_0^{+\infty} g(s)\psi_{xx}(t-s)\,ds + D(\psi) = 0, \end{cases} \tag{1.10}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

in $]0, L[\times]0, +\infty[$, was proved in [20], in both discrete time delay case

$$D(\psi) = \mu_2 \psi_t(t - \tau)$$

and distributed one

$$D(\psi) = \int_0^{+\infty} f(s)\psi_t(t - s)\, ds,$$

where $\mu_2 \in \mathbb{R}^*$ and $f : \mathbb{R}_+ \to \mathbb{R}$ is a given function. In contrast to the situation of absence of delay and/or presence of frictional damping, (1.10) is not necessarily dissipative with respect to its classical energy. To overcome subsequently the difficulties generated by the non-dissipativeness character of (1.10), some new functionals were introduced in [20] to get crucial estimates on some terms generated by the time delay and the infinite memory. The results of [20] generalizes the ones of [18] concerning the particular case $D(\psi) = \mu_2\psi_t(t-\tau)$ and g converges exponentially to zero at infinity.

Similar stability results for various hyperbolic evolution equations with frictional damping and/or memory and/or time delay exist in the literature, in this regard, we refer the reader to [1,3,6,8–10,12,13,16,19,28,34–38,48–52,64–68].

As far as we know, the problem of stability of Timoshenko system with a time delay under infinite memory and/or frictional damping all acting on the transversal displacement has never been treated in the literature. Our goal in this paper is to investigate the effect of each control on the asymptotic behavior of the solutions of (1.1) in the presence of a time delay, and on the decay rate of its energy, when both controls are acting cooperatively, allowing each control to vanish on the whole domain. To our best knowledge, this situation has never been considered before in the literature. Under appropriate assumptions on the history data φ_0, we give an explicit characterization of the decay rate of solutions depending on the growth of g at infinity and the following relations between the weights b and d of, respectively, the frictional damping and time delay:

$$\inf_{[0,L]} (b - |d|) > 0 \tag{1.11}$$

and

$$\inf_{[0,L]} (b - |d|) \leq 0. \tag{1.12}$$

Contrarly to the case (1.11), system (1.1) is not necessarily dissipative with respect to its classical energy when (1.12) holds (see (4.1) and (4.2) below). This creates some difficulties and, so, we prove the exponential stability of (1.1) provided that (1.2) holds, g converges exponetially to zero at infinity and $\|d\|_\infty$ is small enough. In the case when (1.11) holds, we give two general decay estimates (corresponding to the case (1.2) and the opposite one) depending on the smoothness of initial data and growth of g at infinity characterized by the condition (2.9) below introduced in [16]. These results give a generalization of the ones proved by the present author and Messaoudi in [25] concerning the case $d \equiv 0$.

The proof of the well-posedness is based on the maximal monotone operators and semi-group approach (see, for example [33,53]). However, the proof of stability estimates is based on the multiplier method combined with some integral or differential inequalities (see, for example [1,3,10,33–37]) and an approach introduced in [16,19], for a class of abstract hyperbolic systems of single or coupled equations with one infinite memory. In the case when (1.2) does not hold, we use also some ideas given in [3,14,17] to get the decay rate of solutions in terms of the regularity of initial data and the general growth of g at infinity.

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

The paper is organized as follows. In Sect. 2, we set up the hypotheses and present our well-posedness and stability results. In Sect. 3, we prove the well-posedness result. In Sect. 4, we establish some lemmas needed for the proof of the stability results which will be completed in Sect. 5 when (1.2) and (1.11) hold, in Sect. 6 when (1.2) and (1.12) hold, and in Sect. 7 when (1.2) does not hold and (1.11) holds. Finally, some general comments and issues will be given in Sect. 8.

2 Preliminaries and obtained results

2.1 Hypotheses

We consider the following hypotheses:

(H1) The functions $a, b : [0, L] \to \mathbb{R}_+$ and $d : [0, L] \to \mathbb{R}$ are such that

$$a \in C^1([0, L]), \;\; b, \; d \in C([0, L]), \tag{2.1}$$

$$\inf_{[0,L]} (a + b) > 0, \tag{2.2}$$

$$a \equiv 0 \quad \text{or} \quad \inf_{[0,L]} a > 0. \tag{2.3}$$

(H2) The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing of class $C^1(\mathbb{R}_+)$ such that $g(0) > 0$ and

$$g_0 \|a\|_\infty < \frac{k_1 k_2}{k_0 k_1 + k_2}, \tag{2.4}$$

where $g_0 = \int_0^{+\infty} g(s)\, ds$ and k_0 is the smallest constant depending only on L and satisfying (Poincaré's inequality)

$$\int_0^L v^2 dx \leq k_0 \int_0^L v_x^2 \, dx, \quad \forall v \in H_*^1(]0, L[) \tag{2.5}$$

with

$$H_*^1(]0, L[) = \left\{ v \in H^1(]0, L[), \; \int_0^L v \, dx = 0 \right\}. \tag{2.6}$$

(H3) There exist a positive constant α and an increasing strictly convex function G : $\mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty \tag{2.7}$$

such that

$$g'(t) \leq -\alpha g(t), \quad \forall t \geq 0 \tag{2.8}$$

or

$$\int_0^{+\infty} \frac{g(t)}{G^{-1}(-g'(t))} dt + \sup_{t \in \mathbb{R}_+} \frac{g(t)}{G^{-1}(-g'(t))} < +\infty. \tag{2.9}$$

Remark 2.1 1. The hypothesis (2.9) was introduced in [16] and it allows a wider class of relaxation functions than the ones considered in [14,43] (see examples given in [16,26]).

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

2. As in [25], using the second equation and boundary conditions in (1.1), we easily verify that

$$\partial_{tt}\left(\int_0^L \psi\,dx\right) + \frac{k_1}{\rho_2}\int_0^L \psi\,dx = 0.$$

By solving this ordinary differential equation and using the initial data of ψ, we find

$$\int_0^L \psi\,dx = \left(\int_0^L \psi_0\,dx\right)\cos\left(\sqrt{\frac{k_1}{\rho_2}}t\right) + \sqrt{\frac{\rho_2}{k_1}}\left(\int_0^L \psi_1\,dx\right)\sin\left(\sqrt{\frac{k_1}{\rho_2}}t\right). \tag{2.10}$$

Let

$$\tilde{\psi} = \psi - \frac{1}{L}\left(\int_0^L \psi_0\,dx\right)\cos\left(\sqrt{\frac{k_1}{\rho_2}}t\right) - \frac{1}{L}\sqrt{\frac{\rho_2}{k_1}}\left(\int_0^L \psi_1\,dx\right)\sin\left(\sqrt{\frac{k_1}{\rho_2}}t\right). \tag{2.11}$$

Then, one can easily check that

$$\int_0^L \tilde{\psi}\,dx = 0, \tag{2.12}$$

and, hence, Poincaré's inequality (2.5) is applicable for $\tilde{\psi}$ provided that $\tilde{\psi} \in H^1(]0, L[)$. In addition, $(\varphi, \tilde{\psi})$ satisfies (1.1) with initial data

$$\tilde{\psi}_0 = \psi_0 - \frac{1}{L}\int_0^L \psi_0\,dx \quad \text{and} \quad \tilde{\psi}_1 = \psi_1 - \frac{1}{L}\int_0^L \psi_1\,dx$$

instead of ψ_0 and ψ_1, respectively. In the sequel, we work with $\tilde{\psi}$ instead of ψ, but, for simplicity of notation, we use ψ instead of $\tilde{\psi}$.

3. Thanks to Poincaré's inequality (2.5) (applied for $\psi \in H_*^1(]0, L[)$), we have

$$k_1\int_0^L (\varphi_x + \psi)^2\,dx \geq k_1(1-\hat{\epsilon})\int_0^L \varphi_x^2\,dx + k_0k_1\left(1 - \frac{1}{\hat{\epsilon}}\right)\int_0^L \psi_x^2\,dx, \tag{2.13}$$

for any $0 < \hat{\epsilon} < 1$. Then, according to (2.4), we can choose $\hat{\epsilon} > 0$ such that

$$\frac{k_0k_1}{k_0k_1 + k_2} < \hat{\epsilon} < \frac{1}{k_1}(k_1 - g_0\|a\|_\infty)$$

and deduce from (2.13) that

$$\hat{k}\int_0^L \left(\varphi_x^2 + \psi_x^2\right)dx \leq \int_0^L \left(-g_0\|a\|_\infty\varphi_x^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi)^2\right)dx, \tag{2.14}$$

where $\hat{k} = \min\left\{k_1(1-\hat{\epsilon}) - g_0\|a\|_\infty, k_2 + k_0k_1\left(1 - \frac{1}{\hat{\epsilon}}\right)\right\} > 0$.

Because $\int_0^L \varphi_x^2\,dx$ and $\int_0^L \psi_x^2\,dx$ define norms, for φ and ψ on $H_0^1(]0, L[)$ and $H_*^1(]0, L[)$, respectively, then

$$\int_0^L \left(-g_0\|a\|_\infty\varphi_x^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi)^2\right)dx$$

defines a norm on $H_0^1(]0, L[) \times H_*^1(]0, L[)$, for (φ, ψ), equivalent to the one induced by $(H^1(]0, L[))^2$.

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

2.2 Well-posedness

We give here a brief idea about the formulation of (1.1) into an abstract first order system and the related existence, uniqueness and smoothness of solution. Following the ideas of [11,48], let

$$\eta(x,t,s) = \varphi(x,t) - \varphi(x,t-s), \quad \text{for} \quad (x,t,s) \in]0,L[\times]0,+\infty[\times]0,+\infty[\tag{2.15}$$

and

$$z(x,t,p) = \varphi_t(x,t-\tau p), \quad \text{for} \quad (x,t,p) \in]0,L[\times]0,+\infty[\times]0,1[. \tag{2.16}$$

Then

$$\begin{cases} \eta_t + \eta_s - \varphi_t = 0, & \text{in }]0,L[\times]0,+\infty[\times]0,+\infty[, \\ \eta(0,t,s) = \eta(L,t,s) = 0, & \text{in }]0,+\infty[\times]0,+\infty[, \\ \eta(x,t,0) = 0, & \text{in }]0,L[\times]0,+\infty[, \end{cases} \tag{2.17}$$

$$\begin{cases} \tau z_t + z_p = 0, & \text{in }]0,L[\times]0,+\infty[\times]0,1[, \\ z(x,t,0) = \varphi_t(x,t), & \text{in }]0,L[\times]0,+\infty[, \\ z(x,t,1) = \varphi_t(x,t-\tau), & \text{in }]0,L[\times]0,+\infty[\end{cases} \tag{2.18}$$

and

$$\begin{cases} \eta_0(x,s) := \eta(x,0,s) = \varphi_0(x,0) - \varphi_0(x,s), & \text{in }]0,L[\times]0,+\infty[, \\ z_0(x,p) := z(x,0,p) = f_0(x,-\tau p), & \text{in }]0,L[\times]0,1[. \end{cases}$$

Let

$$U = (\varphi, \psi, \varphi_t, \psi_t, z, \eta)^T, \tag{2.19}$$

$$U_0 = (\varphi_0(\cdot,0), \psi_0, \varphi_1, \psi_1, z_0, \eta_0)^T \tag{2.20}$$

and

$$\mathcal{H} = H_0^1(]0,L[) \times H_*^1(]0,L[) \times L^2(]0,L[) \times L_*^2(]0,L[) \times L_\xi \times L_g, \tag{2.21}$$

where

$$L_*^2(]0,L[) = \left\{ v \in L^2(]0,L[), \int_0^L v\,dx = 0 \right\}, \tag{2.22}$$

$$L_g = \left\{ v : \mathbb{R}_+ \to H_0^1(]0,L[), \int_0^L a \int_0^{+\infty} g(s) v_x^2(s)\,ds\,dx < +\infty \right\}, \tag{2.23}$$

$$L_\xi = \left\{ v :]0,1[\to L^2(]0,L[), \int_0^L \xi \int_0^1 v^2(p)\,dp\,dx < +\infty \right\} \tag{2.24}$$

and $\xi : [0,L] \to \mathbb{R}_+$ defined by

$$\xi = \begin{cases} \tau b & \text{if (1.11) holds and } d \neq 0, \\ \tau \|d\|_\infty & \text{if (1.12) holds or } d \equiv 0. \end{cases} \tag{2.25}$$

The spaces L_g and L_ξ endowed with the inner products

$$\langle v, w \rangle_{L_g} = \int_0^L a \int_0^{+\infty} g(s) v_x(s) w_x(s)\,ds\,dx$$

and

$$\langle v, w\rangle_{L_\xi} = \int_0^L \xi \int_0^1 v(p)w(p)\,dp\,dx$$

are Hilbert spaces by vertue of the following Poincaré's inequality:

$$\exists\, \tilde{k}_0 > 0: \quad \int_0^L v^2 dx \leq \tilde{k}_0 \int_0^L v_x^2\, dx, \quad \forall v \in H_0^1(]0, L[) \tag{2.26}$$

and the fact that $a > 0$ if $a \neq 0$ (according to (2.3)), and $\xi > 0$ if $d \neq 0$ (by vertue of (2.25)). The space $\mathcal{H}$ is equipped with the inner product defined by

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} &= \langle v_6, w_6\rangle_{L_g} + \langle v_5, w_5\rangle_{L_\xi} + k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)\,dx \\ &\quad + \int_0^L (-g_0 a \partial_x v_1 \partial_x w_1 + k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4)\; dx,\end{aligned}$$

for any $V = (v_1, \ldots, v_6)^T \in \mathcal{H}$ and $W = (w_1, \ldots, w_6)^T \in \mathcal{H}$. Because L_g and L_ξ are Hilbert spaces, then also $\mathcal{H}$ is a Hilbert space according to (2.14).

Now, we define the linear operators B and A by

$$B(v_1, \ldots, v_6)^T = -\frac{\xi_0}{\rho_1}(0, 0, v_3, 0, 0, 0)^T, \tag{2.27}$$

where

$$\xi_0 = \begin{cases} 0 & \text{if (1.11) holds,} \\ \|d\|_\infty & \text{if (1.12) holds} \end{cases} \tag{2.28}$$

and

$$A\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \begin{pmatrix} -v_3 \\ -v_4 \\ -\frac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \frac{g_0}{\rho_1}\partial_x(a\partial_x v_1) + \frac{b+\xi_0}{\rho_1} v_3 + \frac{d}{\rho_1} v_5(1) - \frac{1}{\rho_1}\int_0^{+\infty} g(s)\partial_x(a\partial_x v_6(s))\,ds \\ -\frac{k_2}{\rho_2}\partial_{xx} v_2 + \frac{k_1}{\rho_2}(\partial_x v_1 + v_2) \\ \frac{1}{\tau}\partial_p v_5 \\ -v_3 + \partial_s v_6 \end{pmatrix}.$$

The system (1.1) is equivalent to

$$\begin{cases} U'(t) + (A + B)U(t) = 0 & \text{on }]0, +\infty[, \\ U(0) = U_0. \end{cases} \tag{2.29}$$

The domain of B is given by $D(B) = \mathcal{H}$. However, the domain of A is defined by

$$D(A) = \left\{V = (v_1, \ldots, v_6)^T \in \mathcal{H},\, AV \in \mathcal{H},\; \partial_x v_2(0) = \partial_x v_2(L) = 0,\; v_5(0) = v_3,\; v_6(0) = 0\right\}$$

and it can be characterized by

$$\begin{aligned} D(A) = \Big\{&(v_1, \ldots, v_6)^T \in H_0^1(]0, L[) \times \left(H^2(]0, L[) \cap H_*^1(]0, L[)\right) \times H_0^1(]0, L[) \\ &\times H_*^1(]0, L[) \times L_\xi \times L_g,\, k_1\partial_{xx} v_1 - g_0\partial_x(a\partial_x v_1) + \int_0^{+\infty} g(s)\partial_x(a\partial_x v_6(s))\,ds \in L^2(]0, L[), \\ &\partial_p v_5 \in L_\xi,\; \partial_s v_6 \in L_g,\; \partial_x v_2(0) = \partial_x v_2(L) = 0,\; v_5(0) = v_3,\; v_6(0) = 0\Big\}. \end{aligned}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

We use the classical notation $D(A^0) = \mathcal{H}$, $D(A^1) = D(A)$ and

$$D(A^n) = \left\{V \in D(A^{n-1}),\ AV \in D(A^{n-1})\right\}, \quad \text{for} \quad n = 2, 3, \ldots,$$

endowed with the graph norm $\|V\|_{D(A^n)} = \sum_{k=0}^{n} \|A^k V\|_{\mathcal{H}}$.

Remark 2.2 If $a \equiv 0$ (resp. $d \equiv 0$), the variable η (resp. z) is not considered, and therefore, the corresponding components in the definition of $U, U_0, \mathcal{H}, B, A$ and $D(A)$ will not appear.

Our well-posedness result reads as follows:

Theorem 2.3 *Assume that (H1)–(H3) are satisfied. For any $n \in \mathbb{N}$ and $U_0 \in D(A^n)$, the system (2.29) has a unique solution*

$$U \in \cap_{k=0}^{n} C^{n-k}(\mathbb{R}_+; D(A^k)). \tag{2.30}$$

2.3 Stability

The energy functional associated with (1.1) is defined by

$$\begin{aligned} E(t) &:= \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2 \\ &= \frac{1}{2}(g \circ \varphi_x)(t) + \frac{1}{2}\int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p)\, dp\, dx \\ &\quad + \frac{1}{2}\int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1(\varphi_x + \psi)^2 + k_2 \psi_x^2 - g_0 a \varphi_x^2\right) dx, \end{aligned} \tag{2.31}$$

where

$$(\phi \circ v)(t) = \int_0^L a \int_0^{+\infty} \phi(s)(v(t) - v(t-s))^2\, ds\, dx, \tag{2.32}$$

for any $v : \mathbb{R} \to L^2(]0, L[)$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$.

Now, we give our first stability result which concerns the case when (1.2) and (1.11) hold.

Theorem 2.4 *Assume that (1.2), (1.11) and (H1)–(H3) are satisfied and let $U_0 \in \mathcal{H}$ such that*

$$\sup_{t \in \mathbb{R}_+} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \varphi_{0x}^2(s - t)\, dx\, ds < +\infty. \tag{2.33}$$

Then there exist positive constants ϵ_0, α_1 and α_2, for which E satisfies

$$E(t) \le \alpha_1 \tilde{G}^{-1}(\alpha_2 t), \quad \forall t \in \mathbb{R}_+, \tag{2.34}$$

where $\tilde{G}(t) = \int_t^1 \frac{1}{G_0(s)}\, ds$ and

$$G_0(s) = \begin{cases} s & \text{if (2.8) holds}, \\ sG'(\epsilon_0 s) & \text{if (2.9) holds}. \end{cases} \tag{2.35}$$

Remark 2.5 1. Because $\lim_{t \to 0^+} \tilde{G}(t) = +\infty$ (by vertue of (H3)), then (2.34) implies that

$$\lim_{t \to +\infty} E(t) = 0. \tag{2.36}$$

2. In case (2.8), $\tilde{G}(s) = -ln\, s$ and (2.34) is reduced to

$$E(t) \le \alpha_1 e^{-\alpha_2 t}, \quad \forall t \in \mathbb{R}_+, \tag{2.37}$$

which is the best decay rate given by (2.34). For specific examples of decay rates given by (2.34), see [17].

Our second stability result concerns the case when (1.2) and (1.12) hold.

Theorem 2.6 *Assume that* (1.2), (1.12), (*H1*) *and* (*H2*) *are satisfied and*

$$\inf_{[0,L]} a > 0 \quad and \quad (2.8)\ holds. \tag{2.38}$$

Then there exists a positive constant d_0 independent of d such that, if

$$\|d\|_\infty^2 + \|d\|_\infty < d_0, \tag{2.39}$$

then, for any $U_0 \in \mathcal{H}$, there exist positive constants α_1 and α_2, for which E satisfies (2.37).

When (1.2) does not hold and (1.11) holds, we prove the following stability result:

Theorem 2.7 *Assume that* (1.11) *and* (*H1*)–(*H3*) *are satisfied. Let $n \in \mathbb{N}^*$ and $U_0 \in D(A^n)$ such that*

$$\sup_{t\in\mathbb{R}_+} \max_{k=0,\dots,n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \left(\frac{\partial^k \varphi_{0x}(s-t)}{\partial s^k}\right)^2 dx\, ds < +\infty. \tag{2.40}$$

Then there exist positive constant ϵ_0 and c_n such that E satisfies

$$E(t) \le G_n\left(\frac{c_n}{t}\right), \quad \forall t > 0, \tag{2.41}$$

where $G_m(s) = G_1(sG_{m-1}(s))$, for $m = 2, \dots, n$ and $s \in \mathbb{R}_+$, $G_1 = G_0^{-1}$ and G_0 is defined in (2.35).

Remark 2.8 When (2.8) holds, $G_n(s) = s^n$ and (2.41) becomes

$$E(t) \le \frac{c_n}{t^n}, \quad \forall t > 0, \tag{2.42}$$

which is the best decay rate given by (2.41). For specific examples of decay rates given by (2.41), see [19].

3 Well-posedness

The proof of Theorem 2.3 is based on the semigroup appraoch by proving that $A+B$ generates a C_0-semigroup in $\mathcal{H}$. We consider the case $\inf_{[0,L]} a > 0$ and $d \neq 0$; the proof in cases $a \equiv 0$ and/or $d \equiv 0$ is similar and simpler.

First, we prove that $-A$ is dissipative. Let $V = (v_1, \dots, v_6)^T \in D(A)$. Exploiting the definition of $D(A)$ and integrating by parts, we find

$$\begin{aligned}\langle -AV, V\rangle_{\mathcal{H}} = &-\frac{1}{2}\int_0^L a \int_0^{+\infty} g(s)\partial_s(\partial_x v_6(s))^2\, ds\, dx - \frac{1}{2\tau}\int_0^L \xi \int_0^1 \partial_p (v_5(p))^2\, dp\, dx \\ &- \int_0^L (b+\xi_0)v_3^2\, dx - \int_0^L d v_3 v_5(1)\, dx.\end{aligned}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Integrating by parts for the first two terms of the above equality, using Young's inequality

$$\lambda_1\lambda_2 \le \frac{\lambda}{2}\lambda_1^2 + \frac{1}{2\lambda}\lambda_2^2, \quad \forall \lambda_1,\ \lambda_2 \in \mathbb{R},\ \forall \lambda > 0 \tag{3.1}$$

(with $\lambda_1 = |v_3|$, $\lambda_2 = |v_5(1)|$ and $\lambda = 1$) and noting that $v_5(0) = v_3$ and $v_6(0) = 0$ (from the definition of $D(A)$), we get

$$\begin{aligned}\langle -AV, V\rangle_{\mathcal{H}} \le\ & \frac{1}{2}\int_0^L a \int_0^{+\infty} g'(s)\,(\partial_x v_6(s))^2\, ds\, dx \\ & + \int_0^L \left(-b - \xi_0 + \frac{\xi}{2\tau} + \frac{|d|}{2}\right) v_3^2\, dx + \int_0^L \left(\frac{|d|}{2} - \frac{\xi}{2\tau}\right) v_5^2(1)\, dx. \end{aligned} \tag{3.2}$$

The definitions (2.25) and (2.28) of ξ and ξ_0 imply that, if (1.11) holds and $d \neq 0$,

$$-b - \xi_0 + \frac{\xi}{2\tau} + \frac{|d|}{2} = \frac{|d|}{2} - \frac{\xi}{2\tau} = \frac{|d| - b}{2} \le 0,$$

and, if (1.12) holds or $d \equiv 0$,

$$-b - \xi_0 + \frac{\xi}{2\tau} + \frac{|d|}{2} = -b + \frac{|d| - \|d\|_\infty}{2} \le 0 \quad \text{and} \quad \frac{|d|}{2} - \frac{\xi}{2\tau} = \frac{|d| - \|d\|_\infty}{2} \le 0.$$

Consequently, the last two integrals in (3.2) are non-positive. Therefore

$$\langle -AV, V\rangle_{\mathcal{H}} \le \frac{1}{2}\int_0^L a \int_0^{+\infty} g'(s)\,(\partial_x v_6(s))^2\, ds\, dx \le 0, \tag{3.3}$$

since g is non-increasing. Then $-A$ is dissipative.

Second, we whow that $Id + A$ is surjective. For this purpose, let $F = (f_1, \ldots, f_6)^T \in \mathcal{H}$, we seek $V = (v_1, \ldots, v_6)^T \in D(A)$ satisfying

$$(Id + A)V = F. \tag{3.4}$$

The first two equations in (3.4) are equivalent to

$$\begin{cases} v_3 = v_1 - f_1, \\ v_4 = v_2 - f_2. \end{cases} \tag{3.5}$$

Using the first equation in (3.5), the last two equations in (3.4) are equivalent to

$$\begin{cases} v_5 + \frac{1}{\tau}\partial_p v_5 = f_5, \\ v_6 + \partial_s v_6 = v_1 - f_1 + f_6, \end{cases} \tag{3.6}$$

then, by solving the ordinary differential equations (3.6) and noting that $v_5(0) = v_3 = v_1 - f_1$ and $v_6(0) = 0$ (see definition of $D(A)$), we get

$$v_5 = \left(v_1 - f_1 + \tau \int_0^p f_5(y) e^{\tau y} dy\right) e^{-\tau p} = e^{-\tau p} v_1 - \left(f_1 - \tau \int_0^p f_5(y) e^{\tau y} dy\right) e^{-\tau p} \tag{3.7}$$

and

$$v_6 = \left(\int_0^s e^y (v_1 - f_1 + f_6(y)) dy\right) e^{-s} = (1 - e^{-s}) v_1 - \left(\int_0^s e^y (f_1 - f_6(y)) dy\right) e^{-s}. \tag{3.8}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

We see that, if

$$(v_1, v_2) \in H_0^1(]0, L[) \times \left(H^2(]0, L[) \cap H_*^1(]0, L[)\right), \tag{3.9}$$

then, from (3.5) to (3.8), we have $(v_3, v_4) \in H_0^1(]0, L[) \times H_*^1(]0, L[)$, $(v_5, v_6) \in L_\xi \times L_g$, $(\partial_p v_5, \partial_s v_6) \in L_\xi \times L_g$, $v_5(0) = v_3$ and $v_6(0) = 0$.

Next, plugging (3.5) and (3.7) into the third and fourth equations in (3.4), we get

$$\begin{cases} \frac{1}{\rho_1}\left(\rho_1 + b + \xi_0 + de^{-\tau}\right) v_1 - \frac{k_1}{\rho_1}\left(\partial_x v_1 + v_2\right)_x \\ \quad + \frac{g_0}{\rho_1}\left(a \partial_x v_1\right)_x - \frac{1}{\rho_1}\displaystyle\int_0^{+\infty} g(s)\left(a \partial_x v_6(s)\right)_x \, ds = f_7, \\ v_2 - \frac{k_2}{\rho_2}\partial_{xx} v_2 + \frac{k_1}{\rho_2}\left(\partial_x v_1 + v_2\right) = f_2 + f_4, \end{cases} \tag{3.10}$$

where

$$f_7 = \frac{1}{\rho_1}(\rho_1 + b + \xi_0 + de^{-\tau}) f_1 + f_3 - \frac{\tau d e^{-\tau}}{\rho_1} \int_0^1 e^{\tau y} f_5(y)\, dy.$$

So, it is sufficient to prove that (3.10), with v_6 given in (3.8), has a solution (v_1, v_2) satisfying (3.9),

$$\partial_x v_2(0) = \partial_x v_2(L) = 0 \tag{3.11}$$

and

$$k_1 \partial_{xx} v_1 - g_0 \partial_x \left(a \partial_x v_1\right) + \int_0^{+\infty} g(s) \partial_x \left(a \partial_x v_6(s)\right) ds \in L^2(]0, L[), \tag{3.12}$$

and then, we replace v_1 and v_2 in (3.5), (3.7) and (3.8) to get $V \in D(A)$ satisfying (3.4). Let (v_1, v_2) satisfying (3.9)–(3.11). By multiplying the equations in (3.10) by $\rho_1 w_1$ and $\rho_2 w_2$, respectively, integrating their sum by parts on $]0, L[$ and exploiting (3.8) and (3.11), we find that (v_1, v_2) is a solution of the system

$$L_1((v_1, v_2), (w_1, w_2)) = L_2(w_1, w_2), \quad \forall (w_1, w_2) \in H_0^1(]0, L[) \times H_*^1(]0, L[), \tag{3.13}$$

where

$$L_1((v_1, v_2), (w_1, w_2)) = \int_0^L \left(k_1 \left(\partial_x v_1 + v_2\right)\left(\partial_x w_1 + w_2\right) + k_2 \partial_x v_2 \partial_x w_2\right) dx,$$

$$+ \int_0^L \left(-a g_1 \partial_x v_1 \partial_x w_1 + \left(\rho_1 + b + \xi_0 + de^{-\tau}\right) v_1 w_1 + \rho_2 v_2 w_2\right) dx,$$

$$L_2((w_1, w_2)) = \int_0^L \left(\rho_1 f_7 w_1 + \partial_x f_8 \partial_x w_1 + \rho_2 (f_2 + f_4) w_2\right) dx,$$

$$g_1 = \int_0^{+\infty} e^{-s} g(s)\, ds \quad \text{and} \quad f_8 = a \int_0^{+\infty} e^{-s} g(s) \int_0^s e^y \left(f_1 - f_6(y)\right) dy\, ds.$$

Since, it is easy to prove that L_1 is a bilinear, continuous and coercive form and L_2 is a linear and continuous form on, respectively,

$$\left(H_0^1(]0, L[) \times H_*^1(]0, L[)\right)^2 \quad \text{and} \quad H_0^1(]0, L[) \times H_*^1(]0, L[)$$

(noting that $g_1 < g_0$ and using (2.14)), so, applying the Lax-Milgram theorem, we deduce that (3.13) admits a unique solution

$$(v_1, v_2) \in H_0^1(]0, L[) \times H_*^1(]0, L[).$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Applying the classical elliptic regularity, it follows that (v_1, v_2) satisfies (3.9)–(3.12). Therefore, the operator $Id + A$ is surjective.

Third, we see that the linear operator B is Lipschitz continuous.

Because $-A$ is dissipative and $Id+A$ is surjective, then A is a maximal monotone operator. Therefore, using Lummer–Phillips theorem (see [53]), we deduce that A is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$. Finally, also $A + B$ is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$ (see [53]: Ch. 3-Theorem 1.1). Consequently, Theorem 2.3 holds from the Hille–Yosida theorem (see [33,53]).

4 Some needed lemmas

We will use c (sometimes c_y, c_{y,y_1}, ..., which depends on some parameters y, y_1, ...), throughout the rest of this paper, to denote a generic positive constant which depends continuously on the initial data U_0 and can be different from step to step, but it does not depend neither on b nor on d.

To get our stability results, we prove first some needed lemmas, for all $U_0 \in D(A)$; so all the calculations are justified. By a simple density arguments ($D(A)$ is dense in $\mathcal{H}$), (2.34) and (2.37) remain valid for any $U_0 \in \mathcal{H}$. The first next seven lemmas, used in [25], are adapted in the present paper to (1.1) by considering the needed modifications related to the presence of delay.

We start by giving the following estimates for the derivative of E:

Lemma 4.1 *The energy functional satisfies, if* (1.11) *holds and* $d \neq 0$,

$$E'(t) \le \frac{1}{2} g' \circ \varphi_x - \frac{1}{2} \inf_{[0,L]} (b - |d|) \int_0^L \varphi_t^2 \, dx, \tag{4.1}$$

and, if (1.12) *holds or* $d \equiv 0$,

$$E'(t) \le \frac{1}{2} g' \circ \varphi_x + \int_0^L (-b + \|d\|_\infty) \varphi_t^2 \, dx. \tag{4.2}$$

Proof By exploiting (2.29), (3.2) and the definition (2.27) of B, we obtain

$$E'(t) \le \frac{1}{2} g' \circ \varphi_x + \int_0^L \left(-b + \frac{\xi}{2\tau} + \frac{|d|}{2}\right) \varphi_t^2 dx + \int_0^L \left(\frac{|d|}{2} - \frac{\xi}{2\tau}\right) \varphi_t^2(t-\tau) dx. \tag{4.3}$$

So, from (2.25), we see that, if (1.11) holds and $d \neq 0$, then

$$-b + \frac{\xi}{2\tau} + \frac{|d|}{2} = \frac{|d|}{2} - \frac{\xi}{2\tau} = -\frac{1}{2}(b - |d|) \le -\frac{1}{2} \inf_{[0,L]} (b - |d|) \le 0.$$

However, if (1.12) holds or $d \equiv 0$, we have

$$-b + \frac{\xi}{2\tau} + \frac{|d|}{2} = -b + \frac{\|d\|_\infty + |d|}{2} \le -b + \|d\|_\infty \quad \text{and} \quad \frac{|d|}{2} - \frac{\xi}{2\tau} = \frac{|d| - \|d\|_\infty}{2} \le 0.$$

Hence, (4.3) yields (4.1) and (4.2). □

Remark 4.2 1. When (1.11) holds, $E' \le 0$, since g is non-increasing, and then (1.1) is dissipative. However, when (1.12) holds, we are unable to determine the sign of E' from (4.2), and therefore, (1.1) is not necessarily dissipative with respect to E at this stage.

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

2. Using the definition of E, (4.1) and (4.2), we see that, for some non-negative constant α_0, $E' \leq \alpha_0 E$. Then, by integrating,

$$E(t) \leq e^{\alpha_0(t-t_0)} E(t_0), \quad \forall t \geq t_0 \geq 0.$$

So, if $E(t_0) = 0$, for some $t_0 \in \mathbb{R}_+$, then $E(t) = 0$, for all $t \geq t_0$, and therefore, (2.34), (2.37) and (2.41) hold. Consequently, without loss of generality, we can assume that $E(t) > 0$, for all $t \in \mathbb{R}_+$.

Lemma 4.3 *The following inequalities hold*:

$$\exists d_1 > 0 : \left(\int_0^L a \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\, ds\, dx\right)^2 \leq d_1 g \circ \varphi_x, \tag{4.4}$$

$$\exists d_2 > 0 : \left(\int_0^L a \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s))\, ds\, dx\right)^2 \leq -d_2 g' \circ \varphi_x, \tag{4.5}$$

$$\exists d_3 > 0 : \left(\int_0^L a' \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s)) ds dx\right)^2 \leq d_3 g \circ \varphi_x. \tag{4.6}$$

$$\left(\int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\, ds\right)^2 \leq g_0 \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))^2 ds, \tag{4.7}$$

$$\left(\int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s)) ds\right)^2 \leq -g(0) \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s))^2 ds. \tag{4.8}$$

Proof If $a \equiv 0$, (4.4)–(4.6) are trivial. If $\inf_{[0,L]} a > 0$, we use the fact that a and a' are bounded and apply Poincaré's and Hölder's inequalities (2.26) (for φ) and

$$\left(\int_0^L |f_1 f_2|\, dx\right)^2 \leq \left(\int_0^L f_1^2\, dx\right)\left(\int_0^L f_2^2\, dx\right), \quad \forall f_1, f_2 \in L^2(]0, L[), \tag{4.9}$$

respectively, to get (4.4)–(4.6). Using again Hölder's inequality (4.9), (4.7) and (4.8) hold. Notice that the constants d_i do not depend neither on b nor on d. □

Lemma 4.4 *The functional*

$$I_1(t) := -\rho_1 \int_0^L a\varphi_t \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\, ds\, dx \tag{4.10}$$

satisfies, for any $\delta > 0$,

$$I_1'(t) \leq -\rho_1 g_0 \int_0^L a\varphi_t^2\, dx + \delta \int_0^L \left(\varphi_t^2 + \varphi_x^2 + \psi_x^2\right) dx$$
$$+ \delta \int_0^L \left(b^2\varphi_t^2 + d^2\varphi_t^2(t-\tau)\right) dx + c\left(1 + \frac{1}{\delta}\right) g \circ \varphi_x - \frac{c}{\delta} g' \circ \varphi_x. \tag{4.11}$$

Proof First, note that

$$\partial_t \left(\int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\, ds\right) = \partial_t \left(\int_{-\infty}^t g(t-s)(\varphi(t) - \varphi(s))\, ds\right)$$
$$= \int_{-\infty}^t g(t-s)\varphi_t(t)\, ds + \int_{-\infty}^t g'(t-s)(\varphi(t) - \varphi(s))\, ds$$
$$= g_0\varphi_t + \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s))\, ds. \tag{4.12}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Then, by differentiating I_1, and using the first equation and the boundary conditions in (1.1), we find

$$
\begin{aligned}
I_1'(t) &= -\rho_1 g_0 \int_0^L a\varphi_t^2\,dx - \rho_1 \int_0^L a\varphi_t \int_0^{+\infty} g'(s)(\varphi(t)-\varphi(t-s))\,ds\,dx \\
&\quad + k_1 \int_0^L a(\varphi_x+\psi) \int_0^{+\infty} g(s)(\varphi_x(t)-\varphi_x(t-s))\,ds\,dx \\
&\quad + \int_0^L a(b\varphi_t + d\varphi_t(t-\tau)) \int_0^{+\infty} g(s)(\varphi(t)-\varphi(t-s))\,ds\,dx \\
&\quad + \int_0^L a^2 \left(\int_0^{+\infty} g(s)(\varphi_x(t)-\varphi_x(t-s))\,ds\right)^2 dx - g_0 \int_0^L a^2\varphi_x \int_0^{+\infty} g(s)(\varphi_x(t) \\
&\quad - \varphi_x(t-s))\,ds\,dx + \int_0^L aa' \left(\int_0^{+\infty} g(s)(\varphi(t)-\varphi(t-s))\,ds\right) \\
&\quad \times \left(\int_0^{+\infty} g(s)(\varphi_x(t)-\varphi_x(t-s))\,ds\right) dx \\
&\quad + k_1 \int_0^L a'(\varphi_x+\psi) \int_0^{+\infty} g(s)(\varphi(t)-\varphi(t-s))\,ds\,dx \\
&\quad - g_0 \int_0^L aa'\varphi_x \int_0^{+\infty} g(s)(\varphi(t)-\varphi(t-s))\,ds\,dx.
\end{aligned}
$$

Therefore, applying Young's and Hölder's inequalities (3.1) and (4.9), for the last eight terms of the above equality, and using (4.4)–(4.7), Poincaré's inequality (2.26), for φ, and the fact that a and a' are bounded, we get (4.11). □

Lemma 4.5 *The functional*

$$
I_2(t) := \int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)\,dx \tag{4.13}
$$

satisfies, for any $\delta > 0$,

$$
\begin{aligned}
I_2'(t) \le{}& \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx - k_1 \int_0^L (\varphi_x+\psi)^2\,dx - k_2 \int_0^L \psi_x^2\,dx \\
&+ g_0 \int_0^L a\varphi_x^2\,dx + \delta \int_0^L \varphi_x^2\,dx + \frac{c}{\delta} \int_0^L \left(b^2\varphi_t^2 + d^2\varphi_t^2(t-\tau)\right) dx + \frac{c}{\delta} g \circ \varphi_x.
\end{aligned} \tag{4.14}
$$

Proof By differentiating I_2, and using the first two equations and boundary conditions in (1.1), we have

$$
\begin{aligned}
I_2'(t) ={}& \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx - k_1 \int_0^L (\varphi_x+\psi)^2\,dx - k_2 \int_0^L \psi_x^2\,dx \\
&+ g_0 \int_0^L a\varphi_x^2\,dx - \int_0^L \varphi(b\varphi_t + d\varphi_t(t-\tau))\,dx \\
&- \int_0^L a\varphi_x \int_0^{+\infty} g(s)(\varphi_x(t)-\varphi_x(t-s))\,ds\,dx.
\end{aligned}
$$

Consequently, aplying Young's and Hölder's inequalities (3.1) and (4.9), for the last two terms of the above equality, and using (4.7), Poincaré's inequality (2.26), for φ, and the fact that a is bounded, we find (4.14). □

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Lemma 4.6 *The functional*

$$I_3(t) := -\rho_2 \int_0^L \psi_t(\varphi_x + \psi)\, dx - \frac{k_2\rho_1}{k_1} \int_0^L \psi_x\varphi_t\, dx$$
$$+ \frac{\rho_2}{k_1} \int_0^L a\psi_t \int_0^{+\infty} g(s)\varphi_x(t-s)\, ds\, dx \tag{4.15}$$

satisfies, for any $\delta,\ \delta_1 > 0$,

$$I_3'(t) \le k_1 \int_0^L (\varphi_x + \psi)^2\, dx - \rho_2 \int_0^L \psi_t^2\, dx$$
$$+ g_0\left(\frac{\delta_1}{2} - 1\right) \int_0^L a\varphi_x^2\, dx + \frac{g_0 k_0 \|a\|_\infty}{2\delta_1} \int_0^L \psi_x^2\, dx$$
$$+ \frac{c}{\delta} \int_0^L \left(b^2\varphi_t^2 + d^2\varphi_t^2(t-\tau)\right) dx$$
$$+ \delta \int_0^L \left(\psi_t^2 + \varphi_x^2 + \psi_x^2\right) dx + \frac{c}{\delta}(g \circ \varphi_x - g' \circ \varphi_x)$$
$$+ \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_{xt}\psi_t\, dx, \tag{4.16}$$

where k_0 *is defined in* (2.5).

Proof Similarly to (4.12) and using that $\lim_{s\to+\infty} g(s) = 0$, we see that

$$\partial_t \left(\int_0^{+\infty} g(s)\varphi_x(t-s)\, ds\right) = \partial_t \left(\int_{-\infty}^t g(t-s)\varphi_x(s)\, ds\right)$$
$$= g(0)\varphi_x + \int_{-\infty}^t g'(t-s)\varphi_x(s)\, ds$$
$$= g(0)\varphi_x + \int_0^{+\infty} g'(s)(\varphi_x(t-s) - \varphi_x(t) + \varphi_x(t))\, ds.$$
$$= -\int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s))\, ds.$$

Therefore, exploiting the first two equations and boundary conditions in (1.1), we have

$$I_3'(t) = k_1 \int_0^L (\varphi_x + \psi)^2\, dx - \rho_2 \int_0^L \psi_t^2\, dx + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \varphi_{xt}\psi_t\, dx$$
$$- g_0 \int_0^L a\varphi_x^2\, dx - g_0 \int_0^L a\varphi_x\psi\, dx + \int_0^L a(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\, ds\, dx$$
$$- \frac{\rho_2}{k_1} \int_0^L a\psi_t \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s))\, ds\, dx + \frac{k_2}{k_1} \int_0^L \psi_x(b\varphi_t + d\varphi_t(t-\tau))\, dx.$$

By applying Young's inequality (3.1), for the last four terms of the above equality, Poincaré's inequality (2.5), for ψ, and using (4.7), (4.8) and the fact that a is bounded, (4.16) is established. □

Now, as in [7], we use a function w to get a crucial estimate.

Lemma 4.7 *The function*

$$w(x,t)=\int_0^x \psi(y,t)\,dy \tag{4.17}$$

satisfies the estimates ($\tilde{k}_0$ is the constant defined in (2.26))

$$\int_0^L w_x^2\,dx=\int_0^L \psi^2\,dx,\quad \forall t\geq 0, \tag{4.18}$$

$$\int_0^L w_t^2\,dx\leq \tilde{k}_0\int_0^L \psi_t^2\,dx,\quad \forall t\geq 0. \tag{4.19}$$

Proof We just have to note that $w_x=\psi$ to get (4.18). On the other hand, using (2.12) (remind that we are working with $\tilde{\psi}$, but we use ψ instead of $\tilde{\psi}$; see Remark 2.1-2),

$$w_t(0,t)=0\quad\text{and}\quad w_t(L,t)=\int_0^L \psi_t(y,t)\,dy=\partial_t\int_0^L \psi(y,t)\,dy=0.$$

Then, applying (4.18) to w_t and using Poincaré's inequality (2.26), for w_t, we arrive at (4.19). □

Lemma 4.8 *The functional*

$$I_4(t):=\rho_1\int_0^L (w\varphi_t+\varphi\varphi_t)\,dx \tag{4.20}$$

satisfies, for any $\delta,\ \epsilon,\ \epsilon_1>0$,

$$\begin{aligned}I_4'(t)\leq&\left(\rho_1+\frac{c_0}{\epsilon}\right)\int_0^L \varphi_t^2\,dx+c_0\epsilon\int_0^L \psi_t^2\,dx\\&+\left(g_0\|a\|_\infty\left(1+\frac{\epsilon_1}{2}\right)-k_1\right)\int_0^L(\varphi_x+\psi)^2\,dx+\frac{g_0k_0\|a\|_\infty}{2\epsilon_1}\int_0^L \psi_x^2\,dx\\&+\delta\int_0^L\left(\varphi_x^2+\psi_x^2\right)\,dx+\frac{c}{\delta}\int_0^L\left(b^2\varphi_t^2+d^2\varphi_t^2(t-\tau)\right)\,dx+\frac{c}{\delta}g\circ\varphi_x,\end{aligned} \tag{4.21}$$

where k_0 is defined in (2.5), $c_0=\frac{\rho_1}{2}\sqrt{\tilde{k}_0}$ and $\tilde{k}_0$ is defined in (2.26).

Proof Using the first two equations and boundary conditions in (1.1), and exploiting the fact that $w(0,t)=w(L,t)=0$ and $w_x=\psi$, we find

$$\begin{aligned}I_4'(t)=&\rho_1\int_0^L \varphi_t^2\,dx-k_1\int_0^L(\varphi_x+\psi)^2\,dx\\&+g_0\int_0^L a(\varphi_x+\psi-\psi)(\varphi_x+\psi)\,dx+\rho_1\int_0^L w_t\varphi_t\,dx\\&-\int_0^L(w+\varphi)(b\varphi_t+d\varphi_t(t-\tau))\,dx-\int_0^L a(\varphi_x+\psi)\int_0^{+\infty}g(s)(\varphi_x(t)\\&-\varphi_x(t-s))\,ds\,dx.\end{aligned}$$

Applying Young's inequality (3.1), for the last four terms of the above equality, Poincaré's inequalities (2.5), for ψ, and (2.26), for φ and w, and exploiting (4.7), (4.18), (4.19) and the fact that a is bounded, we get (4.21). □

We use a functional introduced in [48] (in case $\xi \equiv 1$) to get an estimation on the delay term.

Lemma 4.9 *The functional*

$$I_5(t) = \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_t^2(t-\tau p)\, dp\, dx \tag{4.22}$$

satisfies

$$\begin{aligned} I_5'(t) \le & -2e^{-2\tau} \int_0^L \xi \int_0^1 \varphi_t^2(t-\tau p)\, dp\, dx \\ & + \frac{1}{\tau} \int_0^L \xi \varphi_t^2\, dx - \frac{e^{-2\tau}}{\tau} \int_0^L \xi \varphi_t^2(t-\tau)\, dx. \end{aligned} \tag{4.23}$$

Proof Using (2.16) and the first equation in (2.18), the derivative of I_5 entails

$$\begin{aligned} I_5'(t) &= 2 \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_{tt}(t-\tau p) \varphi_t(x, t-\tau p)\, dp\, dx \\ &= -\frac{2}{\tau} \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_{tp}(t-\tau p) \varphi_t(t-\tau p)\, dp\, dx \\ &= -\frac{1}{\tau} \int_0^L \xi \int_0^1 e^{-2\tau p} \partial_p \varphi_t^2(t-\tau p)\, dp\, dx. \end{aligned}$$

Then, by using an integrating by parts, the above formula can be rewritten as

$$I_5'(t) = -2 \int_0^L \xi \int_0^1 e^{-2\tau p} \varphi_t^2(t-\tau p)\, dp\, dx + \frac{1}{\tau} \int_0^L \xi \varphi_t^2\, dx - \frac{e^{-2\tau}}{\tau} \int_0^L \xi \varphi_t^2(t-\tau)\, dx,$$

which gives (4.23), since $-2e^{-2\tau p} \le -2e^{-2\tau}$, for any $p \in]0, 1[$. □

Let $a_0 := \inf_{[0,L]} a$, $b_0 := \inf_{[0,L]} b$ and, for $N, N_1, N_2, N_3, N_4 \ge 0$,

$$I_6 := NE + N_1 I_1 + N_2 I_2 + I_3 + N_3 I_4 + N_4 I_5. \tag{4.24}$$

Then, by combining (4.11), (4.14), (4.16), (4.21) and (4.23), we obtain

$$\begin{aligned} I_6'(t) \le & -\int_0^L \left(l_0 \varphi_t^2 + l_1 \psi_t^2 + l_2 (\varphi_x + \psi)^2 + l_3 \psi_x^2 \right) dx + l_4 g_0 \int_0^L a \varphi_x^2\, dx + NE'(t) \\ & - 2e^{-2\tau} N_4 \int_0^L \xi \int_0^1 \varphi_t^2(t-\tau p)\, dp\, dx + \delta (N_1 + c_{N_2,N_3}) \int_0^L \left(\varphi_t^2 + \psi_t^2 + \varphi_x^2 + \psi_x^2 \right) dx \\ & - \int_0^L \left(\frac{e^{-2\tau} N_4}{\tau} \xi - \left(\delta N_1 + \frac{c_{N_2,N_3}}{\delta} \right) d^2 \right) \varphi_t^2(t-\tau)\, dx \\ & + \int_0^L \left(\frac{N_4}{\tau} \xi + \left(\delta N_1 + \frac{c_{N_2,N_3}}{\delta} \right) b^2 \right) \varphi_t^2\, dx \\ & + \left(c_{N_1} + \frac{c_{N_1,N_2,N_3}}{\delta} \right) g \circ \varphi_x - \frac{c_{N_1}}{\delta} g' \circ \varphi_x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi_t\, dx, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} l_0 &= N_1 \rho_1 g_0 a_0 - (N_2 + N_3) \rho_1 - \frac{c_0 N_3}{\epsilon}, \\ l_1 &= \rho_2 (1 - N_2) - c_0 \epsilon N_3, \quad l_2 = k_1 (N_2 + N_3 - 1) - g_0 \|a\|_\infty \left(1 + \frac{\epsilon_1}{2} \right) N_3, \\ l_3 &= k_2 N_2 - \frac{g_0 k_0 \|a\|_\infty}{2} \left(\frac{N_3}{\epsilon_1} + \frac{1}{\delta_1} \right) \quad \text{and} \quad l_4 = N_2 + \frac{\delta_1}{2} - 1. \end{aligned}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Now, as in [25], we choose carefully the constants $N,\ N_i,\ \delta,\ \delta_1,\ \epsilon$ and ϵ_1 to get desired signs of l_i.

Case 1 $a \equiv 0$: the second integral in (4.25) drops, $g \circ \varphi_x = g' \circ \varphi_x = 0$ (from the definition (2.32)) and the constants $l_0,\ l_1,\ l_2$ and l_3 do not depent neither on δ_1 nor on ϵ_1. On the other hand,

$$l_0 = -(N_2 + N_3)\rho_1 - \frac{c_0 N_3}{\epsilon} \geq N_1 b_0 - (N_2 + N_3)\rho_1 - \frac{c_0 N_3}{\epsilon} - N_1 b := \tilde{l}_0 - N_1 b,$$

so $\tilde{l}_0 := N_1 b_0 - (N_2 + N_3)\rho_1 - \frac{c_0 N_3}{\epsilon}$. Therefore, we choose

$$N_3 = 1, \quad 0 < N_2 < 1, \quad 0 < \epsilon < \frac{\rho_2}{c_0}(1 - N_2) \quad \text{and} \quad N_1 > \frac{1}{b_0}(N_2 + N_3) + \frac{c_0 N_3}{\epsilon b_0}.$$

Notice that $N_3,\ N_2$ and ϵ do not depend neither on b nor on d. Moreover, because $b_0 > 0$ thanks to (2.2) and $a \equiv 0$, N_1 exists and can be taken in the form $N_1 = \frac{c}{b_0}$, and then $\tilde{l}_0$ as well as $l_1,\ l_2$ and l_3 do not depend neither on b nor on d. According to these choices, we get

$$L := \min\left\{\frac{\tilde{l}_0}{\rho_1}, \frac{l_1}{\rho_2}, \frac{l_2}{k_1}, \frac{l_3}{k_2}\right\} > 0,$$

and then, using (2.14) and (4.25),

$$\begin{aligned} I_6'(t) \leq &-\left(L - c\delta\left(1 + \frac{1}{b_0}\right)\right)\int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1(\varphi_x + \psi)^2\right. \\ &\left.+ k_2 \psi_x^2\right) dx - 2e^{-2\tau} N_4 \int_0^L \xi \int_0^1 \varphi_t^2(t - \tau p)\, dp\, dx \\ &- \int_0^L \left(\frac{e^{-2\tau} N_4}{\tau}\xi - c\left(\frac{\delta}{b_0} + \frac{1}{\delta}\right) d^2\right) \varphi_t^2(t - \tau)\, dx \\ &+ N E'(t) + \int_0^L \left(\frac{N_4}{\tau}\xi + c\left(\frac{b^2\delta}{b_0} + \frac{b^2}{\delta} + \frac{b}{b_0}\right)\right) \varphi_t^2\, dx \\ &+ \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_{xt} \psi_t\, dx. \end{aligned} \tag{4.26}$$

Next, choosing $\delta > 0$ such that

$$L - c\delta\left(1 + \frac{1}{b_0}\right) > 0.$$

Notice that L and c do not depend on δ, b and d; so δ exists and can be taken in the form

$$\delta = \frac{c b_0}{b_0 + 1}, \tag{4.27}$$

and consequently, $L - c\delta\left(1 + \frac{1}{b_0}\right)$ is a positive constant which does not depend neither on b nor on d. At the end, we choose N_4 such that

$$\frac{e^{-2\tau} N_4}{\tau}\xi - c\left(\frac{\delta}{b_0} + \frac{1}{\delta}\right) d^2 \geq 0. \tag{4.28}$$

If $d \equiv 0$, then $\xi \equiv 0$ (thanks to (2.25)), and therefore (4.28) is satisfied, for any $N_4 \geq 0$. Otherwise, if $d \neq 0$, then $\xi = \tau b$ (in vertue of (2.25) and because (1.11) is assumed in this

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

case $a \equiv 0$; see Theorem 2.6), consequently, the choice (4.27) and the inequality $|d| < b$ (according to (1.11)) imply that N_4 can be taken in the form

$$N_4 = \frac{c\|b\|_\infty(b_0+1)}{b_0}. \tag{4.29}$$

Thus, using (2.31), we get from (4.26)

$$\begin{aligned} I_6'(t) &\le -cE_0(t) - \frac{c\|b\|_\infty(b_0+1)}{b_0}E_1(t) + NE'(t) \\ &+ \frac{c\left(\|b\|_\infty(b_0+1)+1\right)}{b_0}\int_0^L b\varphi_t^2\,dx + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_{xt}\psi_t\,dx, \end{aligned} \tag{4.30}$$

where

$$E_0(t) = E(t) - E_1(t) \quad \text{and} \quad E_1(t) = \frac{1}{2}\int_0^L \xi \int_0^1 \varphi_t^2(t-\tau p)\,dp\,dx. \tag{4.31}$$

Case 2. $a_0 > 0$: we choose

$$\epsilon_1 = \frac{k_1 - g_0\|a\|_\infty}{g_0\|a\|_\infty}, \quad \delta_1 = \frac{k_0 g_0\|a\|_\infty}{k_2},$$

$$\frac{k_1\delta_1}{2k_1 - g_0\|a\|_\infty(2+\epsilon_1)} < N_3 < \epsilon_1\left(\frac{k_2(2-\delta_1)}{g_0 k_0\|a\|_\infty} - \frac{1}{\delta_1}\right),$$

$$\max\left\{1 - N_3\left(1 - \frac{g_0\|a\|_\infty(2+\epsilon_1)}{2k_1}\right), \frac{g_0 k_0\|a\|_\infty}{2k_2}\left(\frac{N_3}{\epsilon_1} + \frac{1}{\delta_1}\right)\right\} < N_2 < 1 - \frac{\delta_1}{4},$$

$$0 < \epsilon < \min\left\{\left(2(1-N_2) - \frac{\delta_1}{2}\right)\frac{\rho_2}{c_0 N_3}, \frac{\rho_2(1-N_2)}{c_0 N_3}\right\}$$

and

$$N_1 > \max\left\{\frac{(N_2+N_3)\rho_1 + \frac{c_0 N_3}{\epsilon}}{\rho_1 g_0 a_0}, \frac{\left(2N_2 + N_3 + \frac{\delta_1}{2} - 1\right)\rho_1 + \frac{c_0 N_3}{\epsilon}}{\rho_1 g_0 a_0}\right\}.$$

Notice that ϵ_1 and δ_1 exist and are positive thanks to (2.4) and the property $g_0 > 0$ (because $g(0) > 0$; see (H2)), N_2 exists according to the choice of N_3, ϵ exists from the choice of N_2, and N_1 exists because $\rho_1 g_0 a_0 > 0$. On the other hand, to prove the existence of N_3, we repeat the calculations given in [25]. Using the definitions of ϵ_1 and δ_1, we see that N_3 exists if and only if

$$k_0^2 k_1 (g_0\|a\|_\infty)^3 < k_2(k_2 - k_0 g_0\|a\|_\infty)(k_1 - g_0\|a\|_\infty)^2.$$

Let $y_0 = \frac{k_1 k_2}{k_0 k_1 + k_2}$, $y = g_0\|a\|_\infty \in]0, y_0[$ (see (2.4)) and

$$f(y) = k_0^2 k_1 y^3 - k_2(k_2 - k_0 y)(k_1 - y)^2.$$

We have

$$f'(y) = 3\left(k_0^2 k_1 + k_0 k_2\right)y^2 - 2\left(2k_0 k_1 k_2 + k_2^2\right)y + k_0 k_1^2 k_2 + 2k_1 k_2^2$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

and

$$f''(y) = 6\left(k_0^2 k_1 + k_0 k_2\right) y - 2\left(2k_0 k_1 k_2 + k_2^2\right).$$

Let $y_1 = \frac{2k_0k_1k_2+k_2^2}{3(k_0^2k_1+k_0k_2)}$. We notice that f' is decreasing on $]0, y_1[$, it is increasing on $]y_1, +\infty[$ and

$$f'(y_0) = \frac{k_0^2 k_1^3 k_2 + 2k_0 k_1^2 k_2^2}{k_0 k_1 + k_2} > 0,$$

Moreover, $y_1 \le y_0$ if and only if $k_2 \le k_0 k_1$, and, if $k_2 \le k_0 k_1$,

$$f'(y_1) = \frac{5k_0^2 k_1^2 k_2^2 - k_2^4 + 2k_0 k_1 k_2^3 + 3k_0^3 k_1^3 k_2}{3(k_0^2 k_1 + k_0 k_2)} \ge \frac{9k_2^4}{3(k_0^2 k_1 + k_0 k_2)} > 0.$$

Therefore, f' is positive on $]0, y_0[$, and then $f(y) < f(y_0)$, for any $y \in]0, y_0[$. But $f(y_0) = 0$, hence f is negative on $]0, y_0[$. This guarantees the existence of N_3.

By vertue of these choices, we notice that

$$L := \min\left\{\frac{l_0}{\rho_1}, \frac{l_1}{\rho_2}, \frac{l_2}{k_1}, \frac{l_3}{k_2}\right\} > 0, \quad l_4 \le L,$$

and L does not depend on δ, b and d. Then, using (2.14) and (4.25), we find

$$\begin{aligned}
I_6'(t) \le & -(L - c\delta)\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k_1(\varphi_x+\psi)^2 + k_2\psi_x^2 - ag_0\varphi_x^2\right) dx + NE'(t) \\
& - 2e^{-2\tau} N_4 \int_0^L \xi \int_0^1 \varphi_t^2(t-\tau p)\, dp\, dx - \int_0^L \left(\frac{e^{-2\tau}N_4}{\tau}\xi - c\left(\delta + \frac{1}{\delta}\right) d^2\right)\varphi_t^2(t-\tau)\, dx \\
& + \int_0^L \left(\frac{N_4}{\tau}\xi + c\left(\delta + \frac{1}{\delta}\right) b^2\right)\varphi_t^2\, dx + c\left(1 + \frac{1}{\delta}\right) g\circ\varphi_x - \frac{c}{\delta} g'\circ\varphi_x \\
& + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_{xt}\psi_t\, dx. \qquad (4.32)
\end{aligned}$$

Therefore, choosing $\delta > 0$ and $N_4 \ge 0$ such that $L - c\delta > 0$ and

$$\frac{e^{-2\tau}N_4}{\tau}\xi - c\left(\delta + \frac{1}{\delta}\right) d^2 \ge 0.$$

In vertue of (2.25), N_4 can be chosen in the form $N_4 = c\|d\|_\infty$. Then, using (2.31), (4.30) and (4.32), we find, in both cases $a \equiv 0$ and $a_0 > 0$,

$$\begin{aligned}
I_6'(t) \le & -cE_0(t) - \tilde{c}E_1(t) + NE'(t) \\
& + c\int_0^L \tilde{\xi}\varphi_t^2\, dx + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_{xt}\psi_t\, dx + c(g\circ\varphi_x - g'\circ\varphi_x), \quad (4.33)
\end{aligned}$$

where, thanks to the definition of ξ in case (1.11),

$$\tilde{c} = \begin{cases} \frac{c\|b\|_\infty(b_0+1)}{b_0} & \text{if } a \equiv 0, \\ c\|d\|_\infty & \text{if } a_0 > 0 \end{cases} \qquad (4.34)$$

and

$$\tilde{\xi} = \begin{cases} \frac{\|b\|_\infty(b_0+1)+1}{b_0} b & \text{if } a \equiv 0, \\ \|b\|_\infty b & \text{if } a_0 > 0 \text{ and (1.11) holds}, \\ \|d\|_\infty^2 + \|b\|_\infty b & \text{if } a_0 > 0 \text{ and (1.12) holds}. \end{cases} \qquad (4.35)$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Now, we estimate the term $g \circ \varphi_x$ in (4.33).

Case 1 (2.8) **holds**: then

$$g \circ \varphi_x \leq -\frac{1}{\alpha} g' \circ \varphi_x. \tag{4.36}$$

Case 2 (2.9) **holds**: this case does not concern Theorem 2.6 because of (2.38). For Theorem 2.4 and Theorem 2.7, we apply here an inequality given in [19] (and in [16] in a less general form).

Lemma 4.10 *For any* $\epsilon_0 > 0$, *we have*

$$G'(\epsilon_0 E(t)) g \circ \varphi_x \leq -c g' \circ \varphi_x + c\epsilon_0 E(t) G'(\epsilon_0 E(t)). \tag{4.37}$$

Proof In Theorem 2.4 and Theorem 2.7, it is assumed that (1.11) holds. Then, thanks to (4.1), E is non-increasing. Therefore, the proof is the same as in [19]-Lemma 3.6 (for $B^{\frac{1}{2}} = \partial_x$ and $\|.\| = \|.\|_{L^2(]0,L[)}$). □

Using (4.33), (4.36) and (4.37), we see that, in both two previous cases,

$$\begin{aligned} \frac{G_0(E(t))}{E(t)} I_6'(t) \leq & -\frac{G_0(E(t))}{E(t)} ((c - \tilde{\epsilon}_0) E_0(t) + (\tilde{c} - \tilde{\epsilon}_0) E_1(t)) + N \frac{G_0(E(t))}{E(t)} E'(t) \\ & - c \left(1 + G'(\epsilon_0 E(t))\right) g' \circ \varphi_x + c \frac{G_0(E(t))}{E(t)} \int_0^L \tilde{\xi} \varphi_t^2 \, dx \\ & + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \, dx, \end{aligned} \tag{4.38}$$

where G_0 is defined in (2.35) and

$$\tilde{\epsilon} = \begin{cases} 0 & \text{if (2.8) holds,} \\ c\epsilon_0 & \text{if (2.9) holds.} \end{cases} \tag{4.39}$$

On the other hand, by (2.14) and the definitions of the functionals I_i and E, there exists a positive constant β (not depending on N, b and d) satisfying

$$|N_1 I_1 + N_2 I_2 + I_3 + N_3 I_4 + N_4 I_5| \leq \beta E,$$

which implies that

$$(N - \beta) E \leq I_6 \leq (N + \beta) E. \tag{4.40}$$

Now, at this stage, we distinguish the cases of Theorems 2.4, 2.6 and 2.7.

5 General stability: (1.2) and (1.11) hold

Using (4.1) (in case $d \neq 0$), (4.2) (in case $d \equiv 0$) and the property $g' \leq 0$, we have

$$N E'(t) + c \int_0^L \tilde{\xi} \varphi_t^2 \, dx \leq \int_0^L \left(c\tilde{\xi} - \frac{N}{2} \inf_{[0,L]} (b - |d|)\right) \varphi_t^2 \, dx \tag{5.1}$$

and

$$-g' \circ \varphi_x \leq -2E'(t). \tag{5.2}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Therefore, inserting (5.1) and (5.2) into (4.38), choosing $\epsilon_0 > 0$ such that $\tilde{\epsilon}$ defined in (4.39) satisfies

$$\tilde{\epsilon} < \begin{cases} \min\{c, \tilde{c}\} & \text{if } d \neq 0, \\ c & \text{if } d \equiv 0 \end{cases}$$

(if $d \equiv 0$, then $\xi = E_1 = 0$ and $E = E_0$) and choosing $N \geq 0$ such that

$$c\tilde{\xi} - \frac{N}{2} \inf_{[0,L]} (b - |d|) \leq 0 \quad \text{and} \quad N > \beta$$

(N exists according to (1.11), (4.35) and the boundedness of b), we deduce, from (1.2), (4.38), (4.40) and the fact that $G'(\epsilon_0 E)$ is non-increasing, that $I_6 \sim E$, the last term in (4.38) vanishes and, for some positive constant β_1,

$$\frac{G_0(E(t))}{E(t)} I_6'(t) + cE'(t) \leq -\beta_1 G_0(E(t)). \tag{5.3}$$

Let $\tau_0 > 0$ and

$$F = \tau_0 \left(\frac{G_0(E)}{E} I_6 + cE \right). \tag{5.4}$$

We have $F \sim E$ (because $I_6 \sim E$ and $\frac{G_0(E)}{E}$ is non-increasing) and, using (5.3),

$$F' \leq -\tau_0 \beta_1 G_0(E). \tag{5.5}$$

Then, for $\tau_0 > 0$ such that

$$F \leq E \quad \text{and} \quad F(0) \leq 1, \tag{5.6}$$

we get, for $\alpha_2 = \tau_0 \beta_1 > 0$ (since G_0 is increasing),

$$F' \leq -\alpha_2 G_0(F). \tag{5.7}$$

Then (5.7) implies that

$$(\tilde{G}(F))' \geq \alpha_2, \tag{5.8}$$

where $\tilde{G}(t) = \displaystyle\int_t^1 \frac{1}{G_0(s)}\, ds$. Integrating (5.8) over $[0, t]$ yields

$$\tilde{G}(F(t)) \geq \alpha_2 t + \tilde{G}(F(0)). \tag{5.9}$$

Because $F(0) \leq 1$, $\tilde{G}(1) = 0$ and $\tilde{G}$ is decreasing, we obtain from (5.9) that

$$\tilde{G}(F(t)) \geq \alpha_2 t,$$

which implies that

$$F(t) \leq \tilde{G}^{-1}(\alpha_2 t).$$

The fact that $F \sim E$ gives (2.34). This completes the proof of Theorem 2.4.

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

6 Exponential stability: (1.2) and (1.12) hold

Exploiting (2.38), (4.2), (4.35) and the property $g' \leq 0$, we find

$$NE'(t) + c\int_0^L \tilde{\xi}\varphi_t^2\,dx \leq \int_0^L \left(N(-b + \|d\|_\infty) + c\left(\|d\|_\infty^2 + \|b\|_\infty b\right)\right)\varphi_t^2\,dx$$

$$\leq \int_0^L (c\|b\|_\infty - N)b\varphi_t^2\,dx + \frac{2}{\rho_1}\left(N\|d\|_\infty + c\|d\|_\infty^2\right)E_0(t) \tag{6.1}$$

and

$$-g' \circ \varphi_x \leq -2E'(t) + 2\|d\|_\infty \int_0^L \varphi_t^2\,dx \leq -2E'(t) + \frac{4}{\rho_1}\|d\|_\infty E_0(t). \tag{6.2}$$

Therefore, choosing $N \geq 0$ such that

$$N \geq c\|b\|_\infty \quad \text{and} \quad N > \beta;$$

so $c\|b\|_\infty - N \leq 0$ and $I_6 \sim E$ by vertue of (4.40). The constant N can be choosen in the form

$$N = c(1 + \|b\|_\infty), \tag{6.3}$$

and therefore, inserting (6.1) and (6.2) into (4.38) and noting that the last term in (4.38) vanishes (thanks to (1.2)), $G_0 = Id$ and $\tilde{\epsilon}_0 = 0$ (according to (2.35) and (4.39)), we conclude that, for some positive constant β_2 which does not depend neither on b nor on d,

$$I_6'(t) + cE'(t) \leq -\left(c - \beta_2(1 + \|b\|_\infty)\left(\|d\|_\infty^2 + \|d\|_\infty\right)\right)E_0(t) - \tilde{c}E_1(t).$$

Let $F = I_6 + cE$. The property $I_6 \sim E$ and condition (2.39), for

$$d_0 = \frac{c}{\beta_2(1 + \|b\|_\infty)}, \tag{6.4}$$

lead to $F \sim E$ and

$$F' \leq -\alpha_2 F, \tag{6.5}$$

for some positive constant α_2. By integrating (6.5) over $[0, t]$ and using again the equivalence $F \sim E$, we find (2.37). This ends the proof of Theorem 2.6.

7 Weak stability: (1.2) does not hold and (1.11) holds

In this section, we treat the case when (1.2) does not hold which is more realistic from the physics point of view. We need to estimate the last term in (4.38) using the system (7.1) resulting from differentiating (1.1) with respect to time

$$\begin{cases} \rho_1\varphi_{ttt} - k_1(\varphi_{xt} + \psi_t)_x + d\varphi_{tt}(t-\tau) + b\varphi_{tt} + \displaystyle\int_0^{+\infty} g(s)(a\varphi_{xt}(t-s))_x\,ds = 0, \\ \rho_2\psi_{ttt} - k_2\psi_{xxt} + k_1(\varphi_{xt} + \psi_t) = 0, \\ \varphi_t(0,t) = \psi_{xt}(0,t) = \varphi_t(L,t) = \psi_{xt}(L,t) = 0. \end{cases} \tag{7.1}$$

System (7.1) is well posed for initial data $U_0 \in D(A)$ thanks to Theorem 2.3. Let E_2 be the second-order energy (the energy of (7.1)) defined by

$$E_2(t) = \frac{1}{2}\|U_t(t)\|_{\mathcal{H}}^2. \tag{7.2}$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

A simple calculation (as for (4.1) and (4.2)) implies, in case (1.11), that

$$E_2'(t) \leq \frac{1}{2} g' \circ \varphi_{xt} - \frac{1}{2} \inf_{[0,L]} (b - |d|) \int_0^L \varphi_{tt}^2 \, dx; \tag{7.3}$$

so E_2 is non-increasing (according to (1.11)). Let $\tau_0 = 1$ in (5.4). Thus, similarly to (5.5) (with the same choices of ϵ_0 and N), we deduce from (4.38) that

$$F'(t) \leq -\beta_1 G_0(E(t)) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \, dx. \tag{7.4}$$

Now, as in [25], we use some ideas of [17].

Lemma 7.1 *For any $\epsilon > 0$, we have*

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \, dx \, dt \leq \epsilon \int_S^T G_0(E(t)) \, dt$$
$$+ c_\epsilon \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)), \quad \forall T \geq S \geq 0. \tag{7.5}$$

Proof By integration with respect to t, we get

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \, dx \, dt = \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \left[\frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi \, dx \right]_S^T$$
$$- \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \left(\frac{G_0(E(t))}{E(t)} \right)' \int_0^L \varphi_{xt} \psi \, dx \, dt$$
$$- \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xtt} \psi \, dx \, dt. \tag{7.6}$$

Moreover, applying Poincaré's inequality (2.5), for ψ, and using the definition of E and E_2 and their non-increasingness, we find

$$\left| \left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_0^L \varphi_{xt} \psi \, dx \right| \leq c \, (E(t) + E_2(t))$$
$$\leq c \, (E(S) + E_2(S)), \quad \forall 0 \leq S \leq t.$$

Thus, by integrating by parts the last integral in (7.6) with respect to x and noting that $\frac{G_0(E)}{E}$ is non-increasing, we have

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2 \right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \, dx \, dt$$
$$\leq c \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)) + c \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L |\varphi_{tt}| |\psi_x| dx dt, \; \forall T \geq S \geq 0. \tag{7.7}$$

On the other hand, according to (1.11) and (7.3) (notice also that g is non-incresing), we have

$$\int_0^L \varphi_{tt}^2 \, dx \leq \frac{-2}{\inf_{[0,L]} (b - |d|)} E_2'(t).$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Then, using (2.14) and Young's inequality (3.1), we estimate the last integral in (7.7) as follows:

$$c\int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L |\varphi_{tt}||\psi_x|\,dx\,dt \le \frac{\epsilon\hat{k}}{2}\int_S^T \frac{G_0(E(t))}{E(t)}\psi_x^2\,dx\,dt - c_\epsilon \frac{G_0(E(0))}{E(0)}\int_S^T E_2'(t)\,dt$$
$$\le \epsilon\int_S^T G_0(E(t))\,dt + c_\epsilon \frac{G_0(E(0))}{E(0)} E_2(S), \quad \forall T \ge S \ge 0.$$

Inserting this inequality into (7.7), we get (7.5). □

Now, exploiting (7.4) and (7.5) and choosing $\epsilon \in]0, \beta_1[$, we get, for $\beta_3 = \beta_1 - \epsilon$,

$$\int_S^T F'(t)\,dt \le -\beta_3 \int_S^T G_0(E(t))\,dt + c\frac{G_0(E(0))}{E(0)}\left(E(S) + E_2(S)\right), \quad \forall T \ge S \ge 0. \tag{7.8}$$

By combining (7.8) and the property $F \sim E$, we deduce that, for some positive constant β_4,

$$\int_S^T G_0(E(t))\,dt \le \beta_4\left(1 + \frac{G_0(E(0))}{E(0)}\right)\left(E(S) + E_2(S)\right), \quad \forall T \ge S \ge 0. \tag{7.9}$$

Choosing $S = 0$ in (7.9) and using the fact that $G_0(E)$ is non-increasing, we get

$$G_0(E(T))T \le \int_0^T G_0(E(t))\,dt \le \beta_4\left(1 + \frac{G_0(E(0))}{E(0)}\right)(E(0) + E_2(0)), \quad \forall T \ge 0,$$

which gives (2.41), for $n = 1$, with $c_1 = \beta_4\left(1 + \frac{G_0(E(0))}{E(0)}\right)(E(0) + E_2(0))$, since G_0^{-1} is increasing.

By induction on n, suppose that (2.41) holds and let $U_0 \in D(A^{n+1})$ such that $a \equiv 0$ or (2.8) holds or (2.40) holds, for $n+1$ instead of n. We have $U_t(0) \in D(A^n)$ (thanks to Theorem 2.3) and U_t satisfies the first two equations and the boundary conditions of (1.1). On the other hand, if $a \ne 0$ and (2.8) does not hold, then $U_t(0)$ satisfies (2.40) (because U_0 satisfies (2.40), for $n+1$). Then the energy E_2 of (7.1) (defined in (7.2)) also satisfies, for some positive constant $\tilde{c}_n$,

$$E_2(t) \le G_n\left(\frac{\tilde{c}_n}{t}\right), \quad \forall t > 0. \tag{7.10}$$

Now, choosing $S = \frac{T}{2}$ in (7.9), combining with (2.41) and (7.10), and using the fact that $G_0(E)$ is non-increasing, we deduce that

$$G_0(E(T))T \le 2\int_{\frac{T}{2}}^T G_0(E(t))\,dt \le 2\beta_4\left(1 + \frac{G_0(E(0))}{E(0)}\right)\left(G_n\left(\frac{2c_n}{T}\right) + G_n\left(\frac{2\tilde{c}_n}{T}\right)\right),$$

this implies that, for $c_{n+1} = \max\left\{4\beta_4\left(1 + \frac{G_0(E(0))}{E(0)}\right), 2c_n, 2\tilde{c}_n\right\}$ (notice that G_n is increasing),

$$E(T) \le G_0^{-1}\left(\frac{c_{n+1}}{T}G_n\left(\frac{c_{n+1}}{T}\right)\right) = G_1\left(\frac{c_{n+1}}{T}G_n\left(\frac{c_{n+1}}{T}\right)\right) = G_{n+1}\left(\frac{c_{n+1}}{T}\right).$$

This proves (2.41), for $n+1$. The proof of Theorem 2.7 is completed.

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

8 General comments and issues

We give in this last section some general comments and issues.

Remark 8.1 When (1.2) does not hold and (1.12) holds, proving the stability of (1.1) seems a delicate question (even under smallness condition on $\|d\|_\infty$). In this case, there is a double difficulty: the presence of the last term in (4.38) which can not be absorbed by E itself and the fact that (1.1) and (7.1) are not neccessarily dissipative with respect to E and E_2, respectively (see (4.2) and (7.2)).

Remark 8.2 The regularity $g \in C^1(\mathbb{R}_+)$ can be weaken by assuming that g is differentiable almost everywhere on $\mathbb{R}_+$. On the other hand, our condition (2.9) implies that the set

$$\{s \in \mathbb{R}_+,\ g(s) > 0 \text{ and } g'(0) = 0\} \tag{8.1}$$

is empty. Using the arguments of [64–68], our stability results can be extended to the case of convolution kernels g having flat zones up to a certain extent; that is, the set (8.1) is not negligeable but small enough in some sense.

Remark 8.3 It is interesting to determine the biggest value of d_0 in (2.39) which guarantees the exponential stability (2.37) of (1.1) when (1.2) and (1.12) hold. On the other hand, is the system (1.1) instable when (1.2) and (1.12) hold, but $\|d\|_\infty$ is not small enough?

Remark 8.4 Another interesting question concerns the stability of (1.1) with an additional discrete time delay $\tilde{d}\psi_t(t - \tilde{\tau})$ considered on the second equation, where $\tilde{\tau}$ is a positive constant and $\tilde{d} : [0, L] \to \mathbb{R}$ is a given function.

Remark 8.5 The arguments applied in [20] to get the stability of (1.10) can be adapted to (1.1) and a general stability estimate can be proved when (1.2), (1.12) and (2.9) hold (so g can converge to zero at infinity less faster than exponentially). The arguments of [20] are based on an approach introduced and developped in [64–68]. This approach allowed us to deal with some arbitrary decaying kernels g without assuming explicit conditions on their derivatives g' and to avoid passing by E' in objective to overcome subsequently the difficulties generated by the non-dissipativeness character of (1.10). On the other hand, the arguments of [20] can be used to obtain the stability of (1.1) in case where the discrete time delay $d\varphi_t(t - \tau)$ is replaced by a distributed one

$$\int_0^{+\infty} f(s)\varphi_t(t-s)\,ds,$$

for some given function $f : \mathbb{R}_+ \to \mathbb{R}$. Moreover, the results of the present paper remain true if we replace the linear damping $b\varphi_t$ by a non-linear one $bh(\varphi_t)$, for some given function $h : \mathbb{R} \to \mathbb{R}$. Finally, some other Timoshenko-type systems with controls and time delays on the displacement can be considered (see [25] concerning the case where no delay is considered). To keep away this paper of being too long, we do not discuss these situations.

Remark 8.6 When $\inf_{[0,L]} a > 0$ and $\|d\|_\infty$ is small enough, the stability estimates (2.34) and (2.41) hold true also in case

$$\inf_{[0,L]} (b - |d|) = 0. \tag{8.2}$$

More precisely, we have the following:

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

Theorem 8.7 *Assume that (H1)–(H3) and (8.2) are satisfied and* $\inf_{[0,L]} a > 0$. *Let*

$$\xi = \begin{cases} \tau b & \text{if } d \neq 0, \\ 0 & \text{if } d \equiv 0 \end{cases} \quad \text{and} \quad \xi_0 = 0 \tag{8.3}$$

instead of (2.25) and (2.28). Then the well-posedness result of Theorem 2.3 holds true. Moreover, there exists a positive constant d_0 *independent of d such that, if*

$$\|d\|_\infty < d_0, \tag{8.4}$$

then

1. *Case (1.2) holds: for any* $U_0 \in \mathcal{H}$ *such that (2.8) or (2.33) holds, E satisfies (2.34).*
2. *Case (1.2) does not hold: for any* $n \in \mathbb{N}^*$ *and* $U_0 \in D(A^n)$ *such that (2.8) or (2.40) holds, E satisfies (2.41).*

Proof First, according to (8.2) and (8.3), (2.27) and (3.2) imply that $B \equiv 0$ and (3.3), respectively. The rest of the proof of Theorem 2.3 is identical to the one given in Sect. 3.

Second, under the choice (8.3), (4.3) and (8.2) imply that

$$-g' \circ \varphi_x \leq -2E'(t) \tag{8.5}$$

and

$$E'(t) \leq -\frac{1}{2}\int_0^L b\varphi_t^2\, dx + \frac{\|d\|_\infty}{2}\int_0^L \varphi_t^2\, dx. \tag{8.6}$$

Similarly to (8.5), we have also

$$-g' \circ \varphi_{xt} \leq -2E_2'(t). \tag{8.7}$$

Because $\xi \leq \tau b$, then

$$\tilde{c} = c\|d\|_\infty \quad \text{and} \quad \tilde{\xi} = \|b\|_\infty b \tag{8.8}$$

instead of (4.34) and (4.35). Consequently, using (8.6), we have

$$NE'(t) + c\int_0^L \tilde{\xi}\varphi_t^2\, dx \leq \int_0^L \left(c\tilde{\xi} - \frac{N}{2}b\right)\varphi_t^2\, dx + \frac{N\|d\|_\infty}{\rho_1}E_0(t). \tag{8.9}$$

Therefore, inserting (8.5) and (8.9) into (4.38), we get

$$\begin{aligned} \frac{G_0(E(t))}{E(t)}I_6'(t) \leq & -\frac{G_0(E(t))}{E(t)}\left(\left(c - \frac{N\|d\|_\infty}{\rho_1} - \tilde{\epsilon}_0\right)E_0(t) + (\tilde{c} - \tilde{\epsilon}_0)E_1(t)\right) \\ & - c\left(1 + G'(\epsilon_0 E(t))\right)E'(t) + \frac{G_0(E(t))}{E(t)}\int_0^L \left(c\tilde{\xi} - \frac{N}{2}b\right)\varphi_t^2\, dx \\ & + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\frac{G_0(E(t))}{E(t)}\int_0^L \varphi_{xt}\psi_t\, dx. \end{aligned} \tag{8.10}$$

Choosing $N \geq 0$ such that

$$c\tilde{\xi} - \frac{N}{2}b \leq 0 \quad \text{and} \quad N > \beta;$$

Journal: 13370 Article No.: 0514 TYPESET DISK LE CP Disp.:2017/6/28 Pages: 32 Layout: Small

so N can be taken as in (6.3), therefore $I_6 \sim E$ (due to (4.40)) and, for some positive constant β_5 which does not depend neither on b nor on d (notice that $G'(\epsilon_0 E)$ is non-increasing),

$$\frac{G_0(E(t))}{E(t)} I_6'(t) \leq -\frac{G_0(E(t))}{E(t)} \left((c - \beta_5(1 + \|b\|_\infty)\|d\|_\infty - \tilde{\epsilon}_0)\, E_0(t) + (\tilde{c} - \tilde{\epsilon}_0) E_1(t)\right) - cE'(t) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt}\psi_t\, dx. \tag{8.11}$$

Next, exploiting (8.4), for $d_0 = \frac{c}{\beta_5(1+\|b\|_\infty)}$, and choosing $\epsilon > 0$ such that

$$c - \beta_5(1 + \|b\|_\infty)\|d\|_\infty - \tilde{\epsilon}_0 > 0 \quad \text{and} \quad \tilde{c} - \tilde{\epsilon}_0 > 0,$$

we deduce from (8.11) that, for some positive constant β_6,

$$\frac{G_0(E(t))}{E(t)} I_6'(t) + cE'(t) \leq -\beta_6 G_0(E(t)) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt}\psi_t dx. \tag{8.12}$$

If (1.2) holds, then (8.12) coincides with (5.3) and the proof of (2.34) can be finished as in Sect. 5.

If (1.2) does not hold, we consider the functional F defined in (5.4) with $\tau_0 = 1$, and then (8.12) becomes identical to (7.4). Consequently, the proof of (2.41) can be ended as in Sect. 7. □

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