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Abstract

In this paper, we consider the thermoelastic Bresse system in one-dimensional bounded interval under mixed homogeneous Dirichlet–Neumann boundary conditions and two different kinds of dissipation working only on the longitudinal displacement and given by heat conduction of types I and III. We prove that the exponential stability of the two systems is equivalent to the equality of the three speeds of the wave propagations. Moreover, when at least two speeds of the wave propagations are different, we show the polynomial stability for each system with a decay rate depending on the smoothness of the initial data. The results of this paper complete the ones of Afilal et al. [On the uniform stability for a linear thermoelastic Bresse system with second sound (submitted), 2018], where the dissipation is given by a linear frictional damping or by the heat conduction of second sound. The proof of our results is based on the semigroup theory and a combination of the energy method and the frequency domain approach.

الملخص

في هذا البحث، نعتبر نظام بريس حراري في مجال محدود في بعد واحد وذلك تحت شروط ديريشلت-نيومان الحدية المختلطة والمتجانسة وبوجود نوعين من التبديدات يعملان فقط على الإزاحة الطولية والمعطيان بتوصيل حراري من نوع I وIII. نثبت أن الإستفرار الأسي يكافئ تساوي السرعات الثلاثة لانتشار الأمواج. بالاضافة، نبيّن أنه في حالة اختلاف سرعتين على الأقل، يكون الاستقرار لكل نظام جبريا (كثيري حدود) وذلك بمعدّل تناقص (خمود) يعتمد على ملوسة المعطيات الابتدائية. إن نتائج هذا البحث تكمّل تلك التي أثبتت في [1]، حيث أن التبديد هناك كان نتيجة تخميد احتكاك خطي أو توصيل حراري من نوع الصوت الثاني (صيغة كتانيو). يستند برهان نتائجنا على نظرية شبه الزمر مع دمج لطريقة الطاقة ومقاربة المجالات الترددية

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The effect of the heat conduction of types I and III on the decay rate of the Bresse system via the longitudinal displacement

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1 Introduction

We study in this paper the asymptotic behavior at infinity of the solutions of two coupled systems related to the Bresse model with two different types of dissipation given by heat conduction and working only on the longitudinal displacement. The first system is the Bresse system with thermoelasticity of type I

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t - \beta\theta_{xx} + \delta w_{xt} = 0 & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{1}$$

along with the initial data

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,1), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x) & \text{in } (0,1), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{in } (0,1), \\ \theta(x,0) = \theta_0(x) & \text{in } (0,1) \end{cases} \tag{2}$$

and the mixed homogeneous Dirichlet–Neumann boundary conditions

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0 & \text{in } (0,\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0 & \text{in } (0,\infty). \end{cases} \tag{3}$$

The second system is the Bresse system with thermoelasticity of type III

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_{xt} = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_{tt} - \beta\theta_{xx} - \gamma\theta_{xxt} + \delta w_{xt} = 0 & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{4}$$

along with (2) and (3), and

$$\theta_t(x,0) = \theta_1(x) \quad \text{in } (0,1), \tag{5}$$

where ρ_1, ρ_2, ρ_3, b, k, k_0, δ, β, γ and l are positive constants, w, φ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements, and θ denotes the temperature.

Several well-posedness and stability results for Bresse systems [2] have been obtained during the last few years, where the stability depends on the nature and position of the controls and some relations between the coefficients. Let us mention here some known results concerning the thermoelastic Bresse systems. For more details in what concerns mathematical modeling of the thermoelastic problems, we refer the readers to the works [3,6,7,10,11].

The authors of [13] considered the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) + l\delta\theta = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta q_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_3\theta_t - \theta_{xx} + \beta\left(w_x - l\varphi\right)_t = 0, \\ \rho_3 q_t - q_{xx} + \beta\psi_{xt} = 0 \end{cases} \tag{6}$$

and proved the exponential stability if

$$k - k_0 = \rho_1 b - \rho_2 k = 0, \tag{7}$$

and the polynomial stability in general. In [5], the authors proved that

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0, \\ \rho_3\theta_t - \theta_{xx} + \left(\beta\psi_t\right)_x = 0 \end{cases} \tag{8}$$

is exponentially stable if and only if (7) holds, and it is polynomially stable in general. The results of [5] were generalized in [15] to the case where δ and β are functions of x and vanish on some part of the domain. The authors of [9] proved that the following thermoelastic Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0 \end{cases} \tag{9}$$

is exponentially stable if

$$k - k_0 = \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) - \frac{\tau\delta^2}{b} = 0 \quad \text{and} \quad l \text{ is small,}$$

it is not exponentially stable if

$$k \neq k_0 \quad \text{or} \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) \neq \frac{\tau\delta^2}{b},$$

and it is polynomially stable in general. The author of [4] studied the stability of

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t - \beta\int_0^\infty g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{xt} = 0 & \text{in } (0,1)\times(0,\infty), \end{cases}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function satisfying some hypotheses. He provided a necessary and sufficient condition for exponential stability in terms of the structural parameters of the problem. For particular choices of g, the results of [4] cover the cases of Fourier, Cattaneo and Coleman–Gurtin heat conduction.

For all the above stability results, at least the shear angle displacement ψ was damped via the heat conduction. The authors of [1] considered the Cattaneo heat conduction working only on the longitudinal displacement

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta w_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0 \end{cases} \tag{10}$$

and proved that the exponential stability is equivalent to

$$k\rho_2 - b\rho_1 = (k - k_0)\left(\rho_3 - \frac{\rho_1}{\tau k}\right) - \delta^2 = 0 \tag{11}$$

and

$$l^2 \neq \frac{k_0\rho_2 + b\rho_1}{k_0\rho_2}\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{k\rho_1}{\rho_2\,(k + k_0)}, \quad \forall m \in \mathbb{Z}. \tag{12}$$

Moreover, the polynomial stability of (10) in general was also proved in [1]. Similar stability results were proved in [1] when $\delta\theta_x$ is replaced by δw_t, the last two equations in (10) are neglected and (11) is replaced by (7).

Our objective in this paper is to complete the results of [1] by considering the heat conduction of types I and III. We prove that, when l does not belong to two sequences of real numbers (conditions (15) and (24) below), the exponential stability of the two systems is equivalent to (7). Moreover, we show that the polynomial stability holds in general with two decay rates corresponding to the two cases,

$$\rho_1 b - \rho_2 k = 0 \quad \text{and} \quad \rho_1 b - \rho_2 k \neq 0.$$

The proof of the well-posedness is based on the semigroup theory. However, the stability results are proved using the energy method combined with the frequency domain approach.

The paper is organized as follows. In Sect. 2, we give an idea on the proof of the well-posedness of (1)–(3) and (2)–(5). In Sects. 3 and 4, we prove, respectively, our exponential and polynomial stability results.

2 The semigroup setting

In this section, we give a brief idea on the proof of the well-posedness of (1)–(3) and (2)–(5). We consider the energy space

$$\mathcal{H} = \widetilde{\mathcal{H}} \times \begin{cases} L^2(0,1) & \text{in case (1)}, \\ H_*^1(0,1) \times L^2(0,1) & \text{in case (4)}, \end{cases}$$

where

$$\widetilde{\mathcal{H}} = H_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1),$$

$$H_*^1(0,1) = \left\{f \in H^1(0,1) : f(0) = 0\right\} \quad \text{and} \quad \widetilde{H}_*^1(0,1) = \left\{f \in H^1(0,1) : f(1) = 0\right\}.$$

The space $\mathcal{H}$ is equipped with the inner product

$$\begin{aligned}\langle \Phi_1, \Phi_2\rangle_{\mathcal{H}} &= k\left\langle(\varphi_{1x} + \psi_1 + l\,w_1),(\varphi_{2x} + \psi_2 + l\,w_2)\right\rangle_{L^2(0,1)} + b\,\langle\psi_{1x},\psi_{2x}\rangle_{L^2(0,1)} \\ &\quad + k_0\left\langle(w_{1x} - l\varphi_1),(w_{2x} - l\varphi_2)\right\rangle_{L^2(0,1)} + \rho_1\,\langle\tilde\varphi_1,\tilde\varphi_2\rangle_{L^2(0,1)} + \rho_2\langle\tilde\psi_1,\tilde\psi_2\rangle_{L^2(0,1)} \\ &\quad + \rho_1\,\langle\tilde w_1,\tilde w_2\rangle_{L^2(0,1)} + \begin{cases}\rho_3\,\langle\theta_1,\theta_2\rangle_{L^2(0,1)} & \text{in case (1)}, \\ \beta\,\langle\theta_{1x},\theta_{2x}\rangle_{L^2(0,1)} + \rho_3\left\langle\tilde\theta_1,\tilde\theta_2\right\rangle_{L^2(0,1)} & \text{in case (4)},\end{cases}\end{aligned}$$

where (for $j = 1, 2$)

$$\Phi_j = \begin{cases}(\varphi_j,\,\tilde\varphi_j,\,\psi_j,\,\tilde\psi_j,\,w_j,\,\tilde w_j,\,\theta_j)^T & \text{in case (1)}, \\ (\varphi_j,\,\tilde\varphi_j,\,\psi_j,\,\tilde\psi_j,\,w_j,\,\tilde w_j,\,\theta_j,\,\tilde\theta_j)^T & \text{in case (4)}.\end{cases}$$

We consider also

$$\Phi = \begin{cases}\left(\varphi,\,\tilde\varphi,\,\psi,\,\tilde\psi,\,w,\,\tilde w,\,\theta\right)^T & \text{in case (1)}, \\ \left(\varphi,\,\tilde\varphi,\,\psi,\,\tilde\psi,\,w,\,\tilde w,\,\theta,\,\tilde\theta\right)^T & \text{in case (4)}\end{cases} \tag{13}$$

and

$$\Phi_0 = \begin{cases} (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ w_0,\ w_1,\ \theta_0)^T & \text{in case (1)}, \\ (\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ w_0,\ w_1,\ \theta_0,\ \theta_1)^T & \text{in case (4)}, \end{cases}$$

where

$$\tilde{\varphi} = \varphi_t, \quad \tilde{\psi} = \psi_t, \quad \tilde{w} = w_t \quad \text{and} \quad \tilde{\theta} = \theta_t.$$

Systems (1)–(3) and (2)–(5) can be written as a first-order system given by

$$\begin{cases} \Phi_t = \mathcal{A}\Phi & \text{in } (0, \infty), \\ \Phi\,(t = 0) = \Phi_0, \end{cases} \tag{14}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\Phi = \begin{pmatrix} \tilde{\varphi} \\ \frac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \frac{lk_0}{\rho_1}\left(w_x - l\varphi\right) \\ \tilde{\psi} \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) \\ \tilde{w} \\ \frac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \frac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) - \frac{\delta}{\rho_1}\theta_x \\ \frac{\beta}{\rho_3}\theta_{xx} - \frac{\delta}{\rho_3}\tilde{w}_x \end{pmatrix}$$

in case (1), and

$$\mathcal{A}\Phi = \begin{pmatrix} \tilde{\varphi} \\ \frac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \frac{lk_0}{\rho_1}\left(w_x - l\varphi\right) \\ \tilde{\psi} \\ \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) \\ \tilde{w} \\ \frac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \frac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) - \frac{\delta}{\rho_1}\tilde{\theta}_x \\ \tilde{\theta} \\ \frac{1}{\rho_3}\left(\beta\theta + \gamma\tilde{\theta}\right)_{xx} - \frac{\delta}{\rho_3}\tilde{w}_x \end{pmatrix}$$

in case (4). The domain of $\mathcal{A}$ is defined by

$$D\left(\mathcal{A}\right) = \left\{ \begin{array}{l} \Phi \in \mathcal{H} \mid \varphi,\ \theta \in H_*^2\,(0,1)\,;\ \psi,\ w \in \widetilde{H}_*^2\,(0,1)\,;\ \tilde{\varphi} \in H_*^1\,(0,1)\,; \\ \tilde{\psi},\ \tilde{w} \in \widetilde{H}_*^1\,(0,1)\,;\ \varphi_x\,(1) = \psi_x\,(0) = w_x\,(0) = \theta_x\,(1) = 0 \end{array} \right\}$$

in case (1), and

$$D\left(\mathcal{A}\right) = \left\{ \begin{array}{l} \Phi \in \mathcal{H} \mid \varphi,\ \beta\theta + \gamma\tilde{\theta} \in H_*^2\,(0,1)\,;\ \psi,\ w \in \widetilde{H}_*^2\,(0,1)\,;\ \tilde{\varphi},\ \tilde{\theta} \in H_*^1\,(0,1)\,; \\ \tilde{\psi},\ \tilde{w} \in \widetilde{H}_*^1\,(0,1)\,;\ \varphi_x\,(1) = \psi_x\,(0) = w_x\,(0) = \theta_x\,(1) = 0 \end{array} \right\}$$

in case (4), where

$$H^2_*(0,1) = H^2(0,1) \cap H^1_*(0,1) \quad \text{and} \quad \widetilde{H}^2_*(0,1) = H^2(0,1) \cap \widetilde{H}^1_*(0,1).$$

The following well-posedness results for (14) hold:

Theorem 2.1 *Assume that*

$$l \notin \frac{\pi}{2} + \pi\mathbb{N}. \tag{15}$$

Then, for any $m \in \mathbb{N}$ and $\Phi_0 \in D(\mathcal{A}^m)$, system (14) admits a unique solution

$$\Phi \in \cap_{j=0}^{m} C^{m-j}\left(\mathbb{R}_+; D\left(\mathcal{A}^j\right)\right). \tag{16}$$

Proof First, from the definition of $H^1_*(0,1)$ and $\widetilde{H}^1_*(0,1)$, we see that, if

$$(\varphi, \psi, w) \in H^1_*(0,1) \times \widetilde{H}^1_*(0,1) \times \widetilde{H}^1_*(0,1)$$

satisfies

$$k\,\|(\varphi_x + \psi + l\,w)\|^2_{L^2(0,1)} + b\,\|\psi_x\|^2_{L^2(0,1)} + k_0\,\|(w_x - l\varphi)\|^2_{L^2(0,1)} = 0,$$

then

$$\psi = 0, \quad \varphi = -c\sin(lx) \quad \text{and} \quad w = c\cos(lx),$$

where c is a constant such that

$$c = 0 \quad \text{or} \quad l \in \frac{\pi}{2} + \pi\mathbb{N}.$$

Then condition (15) implies that $\varphi = \psi = w = 0$, and thus, $\mathcal{H}$ is a Hilbert space.

Second, we prove that $\mathcal{A}$ is dissipative. Indeed, using the definition of $\mathcal{A}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, and integrating by parts, we get

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = \begin{cases} -\beta\,\|\theta_x\|^2_{L^2(0,1)} & \text{in case (1)}, \\ -\gamma\,\left\|\tilde{\theta}_x\right\|^2_{L^2(0,1)} & \text{in case (4)}. \end{cases} \tag{17}$$

Hence, $\mathcal{A}$ is dissipative in $\mathcal{H}$.

Third, we show that, for any $F \in \mathcal{H}$, there exists $Z \in D(\mathcal{A})$ satisfying

$$\mathcal{A}Z = F, \tag{18}$$

that is $0 \in \rho(\mathcal{A})$. Let $F = (f_1, \ldots, f_j)^T$ and $Z = (z_1, \ldots, z_j)^T$, where $j = 7$ in case (1), and $j = 8$ in case (4). The first, third and fifth equations in (18) are equivalent to

$$z_2 = f_1, \quad z_4 = f_3 \quad \text{and} \quad z_6 = f_5, \tag{19}$$

and the seventh equation in case (4) becomes

$$z_8 = f_7. \tag{20}$$

So, because $F \in \mathcal{H}$, z_2, z_4, z_6 and z_8 have the required regularity in $D(\mathcal{A})$. Then, the last equation in (18) is reduced to

$$z_{7xx} = \frac{\delta}{\beta} f_{5x} + \frac{\rho_3}{\beta} f_7 \tag{21}$$

in case (1), and

$$(\beta z_7 + \gamma f_7)_{xx} = \delta f_{5x} + \rho_3 f_8 \tag{22}$$

in case (4). By a direct integration, we see that each equation in (21) and (22) has a unique solution z_7 satisfying the needed regularity and Neumann boundary condition in $D(\mathcal{A})$. Therefore, the second, fourth and sixth equations in (18) become

$$\begin{cases} k\,(z_{1x} + z_3 + l\,z_5)_x + lk_0\,(z_{5x} - lz_1) = \rho_1 f_2, \\ bz_{3xx} - k\,(z_{1x} + z_3 + l\,z_5) = \rho_2 f_4, \\ k_0\,(z_{5x} - lz_1)_x - lk\,(z_{1x} + z_3 + l\,z_5) = \tilde{f}, \end{cases} \tag{23}$$

where

$$\tilde{f}=\begin{cases}\delta z_{7x}+\rho_1 f_6 & \text{in case (1)},\\ \delta f_{7x}+\rho_1 f_6 & \text{in case (4)}.\end{cases}$$

To prove that (23) admits a solution (z_1, z_3, z_5) satisfying the required regularity and Neumann boundary condition in $D(\mathcal{A})$, we consider the variational formulation of (23) and use the Lax–Milgram theorem and classical elliptic regularity arguments. So, this proves that (18) has a unique solution $Z \in D(\mathcal{A})$. By the resolvent identity, we have $\lambda I-\mathcal{A}$ is surjective, for any $\lambda>0$ (see [14]). Consequently, the Lumer–Phillips theorem implies that $\mathcal{A}$ is the infinitesimal generator of a linear C_0 semigroup of contractions on $\mathcal{H}$. Finally, Theorem 2.1 holds (see [16]) □

3 Exponential stability

Our objective in this section is to show the following exponential stability result:

Theorem 3.1 *We assume that* (15) *holds. Then the semigroup associated with* (14) *is exponentially stable if and only if*

$$l^2 \neq \frac{\rho_2 k_0+\rho_1 b}{\rho_2 k_0}\left(\frac{\pi}{2}+m\pi\right)^2+\frac{\rho_1 k}{\rho_2\,(k+k_0)}, \quad \forall m\in\mathbb{Z} \tag{24}$$

and

$$k-k_0=\rho_1 b-\rho_2 k=0. \tag{25}$$

The proof is based on the following theorem:

Theorem 3.2 [8,17] *A C_0 semigroup of contractions on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable if and only if*

$$i\,\mathbb{R}\subset\rho(\mathcal{A}) \tag{26}$$

and

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}<\infty. \tag{27}$$

Proof We prove that (24) is equivalent to (26), and (25) is equivalent to (27). So Theorem 3.2 implies Theorem 3.1. □

3.1 Conditions (24) and (26) are equivalent

Note that, according to the fact that $0\in\rho(\mathcal{A})$ (see Sect. 2), $\mathcal{A}^{-1}$ is bounded and it is a bijection between $\mathcal{H}$ and $D(\mathcal{A})$. Since $D(\mathcal{A})$ has a compact embedding into $\mathcal{H}$, so it follows that $\mathcal{A}^{-1}$ is a compact operator, which implies that the spectrum of $\mathcal{A}$ is discrete. Then $i\lambda\in\rho(\mathcal{A})$ if and only if λ is not an eigenvalue of $\mathcal{A}$.

Let $\lambda\in\mathbb{R}^*$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{A}$ by proving that the unique solution $\Phi\in D(\mathcal{A})$ of the equation

$$\mathcal{A}\,\Phi=i\,\lambda\,\Phi \tag{28}$$

is $\Phi=0$. Let Φ be given by (13). The Eq. (28) means that

$$\begin{cases}\tilde{\varphi}=i\lambda\varphi, \quad \tilde{\psi}=i\lambda\psi, \quad \tilde{w}=i\lambda w,\\ \dfrac{k}{\rho_1}(\varphi_x+\psi+l\,w)_x+\dfrac{lk_0}{\rho_1}(w_x-l\varphi)=i\lambda\tilde{\varphi},\\ \dfrac{b}{\rho_2}\psi_{xx}-\dfrac{k}{\rho_2}(\varphi_x+\psi+l\,w)=i\lambda\tilde{\psi},\\ \dfrac{k_0}{\rho_1}(w_x-l\varphi)_x-\dfrac{lk}{\rho_1}(\varphi_x+\psi+l\,w)-\dfrac{\delta}{\rho_1}\theta_x=i\lambda\tilde{w},\\ \dfrac{\beta}{\rho_3}\theta_{xx}-\dfrac{\delta}{\rho_3}\tilde{w}_x=i\lambda\theta\end{cases} \tag{29}$$

in case (1), and

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \quad \tilde{\theta} = i\lambda\theta, \\ \dfrac{k}{\rho_1}\left(\varphi_x + \psi + l\, w\right)_x + \dfrac{lk_0}{\rho_1}\left(w_x - l\varphi\right) = i\lambda\tilde{\varphi}, \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}\left(\varphi_x + \psi + l\, w\right) = i\lambda\tilde{\psi}, \\ \dfrac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \dfrac{lk}{\rho_1}\left(\varphi_x + \psi + l\, w\right) - \dfrac{\delta}{\rho_1}\tilde{\theta}_x = i\lambda\tilde{w}, \\ \dfrac{1}{\rho_3}\left(\beta\theta + \gamma\tilde{\theta}\right)_{xx} - \dfrac{\delta}{\rho_3}\tilde{w}_x = i\lambda\tilde{\theta} \end{cases} \tag{30}$$

in case (4). Using (17) and (28), we find

$$0 = Re\, i\lambda \left\|\Phi\right\|_{\mathcal{H}}^2 = Re\, \langle i\lambda\Phi, \Phi\rangle_{\mathcal{H}} = Re\, \langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = \begin{cases} -\beta \left\|\theta_x\right\|_{L^2(0,1)}^2 & \text{in case (1)}, \\ -\gamma \left\|\tilde{\theta}_x\right\|_{L^2(0,1)}^2 & \text{in case (4)}. \end{cases}$$

Then

$$\begin{cases} \theta_x = 0 & \text{in case (1)}, \\ \tilde{\theta}_x = 0 & \text{in case (4)}. \end{cases} \tag{31}$$

But $\theta,\ \tilde{\theta} \in H_*^1(0,1)$ (since $\Phi \in D(\mathcal{A})$), then, using the Poincaré's inequality, (31) and the fourth equation in (30), we deduce that

$$\begin{cases} \theta = 0 & \text{in case (1)}, \\ \theta = \tilde{\theta} = 0 & \text{in case (4)}. \end{cases} \tag{32}$$

Therefore, from (32) and the third and last equations in (29) and (30), we find

$$w_x = \tilde{w}_x = 0. \tag{33}$$

As $w,\ \tilde{w} \in \widetilde{H}_*^1(0,1)$ and according to Poincaré's inequality, we have

$$w = \tilde{w} = 0. \tag{34}$$

Using (32) and (34), we see that (29) and (30) are reduced to

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \\ \left(l^2k_0 - \rho_1\lambda^2\right)\varphi - k\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\lambda^2\psi - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0, \\ \varphi_x + \psi = -\dfrac{k_0}{k}\varphi_x. \end{cases} \tag{35}$$

Now, we follow the proof given in [1]. By deriving the fifth equation in (35) and combining the third one, we see that

$$\varphi_{xx} + \alpha\varphi = 0, \tag{36}$$

where $\alpha = \frac{l^2k_0 - \rho_1\lambda^2}{k_0}$. We distinguish three cases.

Case 1 $\lambda^2 = \frac{l^2k_0}{\rho_1}$. Then

$$\varphi(x) = c_1x + c_2,$$

for $c_1,\ c_2 \in \mathbb{C}$. Using the boundary conditions

$$\varphi(0) = \varphi_x(1) = 0, \tag{37}$$

we find

$$\varphi = 0, \tag{38}$$

which implies that, using the first two equations and the last one in (35),

$$\widetilde{\varphi} = 0 \tag{39}$$

and

$$\psi = \widetilde{\psi} = 0. \tag{40}$$

Consequently, we get

$$\Phi = 0. \tag{41}$$

Case 2 $\lambda^2 > \frac{l^2 k_0}{\rho_1}$. Then

$$\varphi(x) = c_1 e^{\sqrt{-\alpha}x} + c_2 e^{-\sqrt{-\alpha}x}.$$

Using again the boundary conditions (37), we find (38), and similarly to case 1, we arrive at (41).

Case 3 $\lambda^2 < \frac{l^2 k_0}{\rho_1}$. Then

$$\varphi(x) = c_1 \cos\left(\sqrt{\alpha}x\right) + c_2 \sin\left(\sqrt{\alpha}x\right).$$

Using the boundary conditions (37), we deduce that $c_1 = 0$, and

$$c_2 = 0 \quad \text{or} \quad \exists m \in \mathbb{Z}: \ \alpha = \left(\frac{\pi}{2} + m\pi\right)^2. \tag{42}$$

If $c_2 = 0$, then (38) holds, and as before, we find (41).

If $c_2 \neq 0$, then, by (42), we have

$$\exists m \in \mathbb{Z}: \ \frac{l^2 k_0 - \rho_1 \lambda^2}{k_0} = \left(\frac{\pi}{2} + m\pi\right)^2. \tag{43}$$

Therefore, the fifth equation in (35) is equivalent to

$$\psi(x) = -c_2 \left(1 + \frac{k_0}{k}\right) \sqrt{\alpha} \cos\left(\sqrt{\alpha}x\right), \tag{44}$$

and then the third and fourth equations in (35) are reduced to

$$\lambda^2 = \frac{k_0 \left[k k_0 + b l^2 (k + k_0)\right]}{(k + k_0)(k_0 \rho_2 + b \rho_1)}. \tag{45}$$

We see that (43) and (45) lead to

$$\exists m \in \mathbb{Z}: \ l^2 = \frac{\rho_2 k_0 + \rho_1 b}{\rho_2 k_0} \left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k}{\rho_2 (k + k_0)};$$

that is (24) does not hold. So, if (24) holds, we get a contradiction, and hence, $c_2 = 0$ and, as before, we find (41). If (24) does not hold, then, for $\lambda \in \mathbb{R}$ satisfying (45), the function

$$\Phi(x) = c_2 \left(\sin\left(\sqrt{\alpha}x\right), i\lambda \sin\left(\sqrt{\alpha}x\right), -\left(1 + \frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right), \right.$$
$$\left. -i\lambda \left(1 + \frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right), 0, 0, 0, 0\right)^T$$

is a solution of (28), for any $c_2 \in \mathbb{C}$, and then λ is an eigenvalue of $\mathcal{A}$. Finally, (26) holds if and only if (24) holds.

3.2 Condition (25) implies (27)

We assume that (25) holds and prove (27). Let us proceed by contradiction. So, we assume that (27) is false, then there exist sequences $(\Phi_n)_n \subset \mathcal{D}(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}} = 1, \quad \forall n \in \mathbb{N}, \tag{46}$$

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{47}$$

and

$$\lim_{n\to\infty} \|(i\,\lambda_n\, I - \mathcal{A})\, \Phi_n\|_{\mathcal{H}} = 0. \tag{48}$$

3.2.1 Case of system (1)

The limit (48) implies the following ones:

$$\begin{cases} i\lambda_n\varphi_n - \widetilde{\varphi}_n \longrightarrow 0 \ \text{ in } H^1_*(0,1)\,, \\ i\lambda_n\rho_1\widetilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + lw_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) \longrightarrow 0 \ \text{ in } L^2(0,1)\,, \\ i\lambda_n\psi_n - \widetilde{\psi}_n \longrightarrow 0 \text{ in } \widetilde{H}^1_*(0,1)\,, \\ i\lambda_n\rho_2\widetilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + lw_n\right) \longrightarrow 0 \ \text{ in } L^2(0,1)\,, \\ i\lambda_n w_n - \widetilde{w}_n \longrightarrow 0 \ \text{ in } \widetilde{H}^1_*(0,1)\,, \\ i\lambda_n\rho_1\widetilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + lw_n\right) + \delta\theta_{nx} \longrightarrow 0 \ \text{ in } L^2(0,1)\,, \\ i\lambda_n\rho_3\theta_n - \beta\theta_{nxx} + \delta\widetilde{w}_{nx} \longrightarrow 0 \ \text{ in } L^2(0,1)\,. \end{cases} \tag{49}$$

We will arrive to a contradiction with (46) by proving that

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = 0. \tag{50}$$

Some of the calculations below are used in [1].

Estimate on θ_n Taking the inner product of $(i\,\lambda_n\, I - \mathcal{A})\,\Phi_n$ with Φ_n in $\mathcal{H}$ and using (17), we get

$$Re\,\langle (i\,\lambda_n\, I - \mathcal{A})\,\Phi_n, \Phi_n\rangle_{\mathcal{H}} = \beta\,\|\theta_{nx}\|^2_{L^2(0,1)}\,. \tag{51}$$

Using (46) and (48), we deduce that

$$\theta_{nx} \longrightarrow 0 \ \text{ in } L^2(0,1)\,. \tag{52}$$

Because $\theta_n(0) = 0$, then we get from (52) that

$$\theta_n \longrightarrow 0 \ \text{ in } L^2(0,1)\,. \tag{53}$$

Estimates on φ_n**,** ψ_n **and** w_n Multiplying $(49)_1$, $(49)_3$ and $(49)_5$ by $\frac{1}{\lambda_n}$, and using (46) and (47), we find

$$\begin{cases} \varphi_n \longrightarrow 0 \ \text{ in } L^2(0,1)\,, \\ \psi_n \longrightarrow 0 \ \text{ in } L^2(0,1)\,, \\ w_n \longrightarrow 0 \ \text{ in } L^2(0,1)\,. \end{cases} \tag{54}$$

Estimate on $\frac{1}{\lambda_n} w_{nxx}$ Applying the triangle inequality, we have

$$\left\|\frac{w_{nxx}}{\lambda_n}\right\|_{L^2(0,1)} \leq \frac{1}{k_0\,|\lambda_n|}\left\|i\lambda_n\rho_1\widetilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + lw_n\right) + \delta\theta_{nx}\right\|_{L^2(0,1)}$$
$$+\frac{1}{k_0}\left\|i\rho_1\widetilde{w}_n + \frac{lk_0}{\lambda_n}\varphi_{nx} + \frac{lk}{\lambda_n}\left(\varphi_{nx} + \psi_n + lw_n\right) + \delta\frac{\theta_{nx}}{\lambda_n}\right\|_{L^2(0,1)}.$$

Then, by (46), (47), $(49)_6$ and (52), we deduce that

$$\left(\frac{1}{\lambda_n} w_{nxx}\right)_n \text{ is bounded in } L^2(0,1). \tag{55}$$

Estimates on w_{nx}, $\frac{1}{\lambda_n}\widetilde{w}_{nx}$ and $\frac{1}{\lambda_n}\widetilde{w}_n$ Taking the inner product of $(49)_7$ with $\dfrac{i w_{nx}}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\rho_3 \langle \theta_n, w_{nx}\rangle_{L^2(0,1)} + \beta \left\langle \theta_{nx}, \frac{i w_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)} - \delta \left\langle \left(i\lambda_n w_{nx} - \widetilde{w}_{nx}\right), \frac{i w_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)}$$
$$+\delta \left\| w_{nx}\right\|^2_{L^2(0,1)} \longrightarrow 0.$$

Using (46), (47), $(49)_5$, (52), (53) and (55), we deduce that

$$w_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1), \tag{56}$$

and from $(49)_5$, we have

$$\frac{1}{\lambda_n}\widetilde{w}_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{57}$$

As $\widetilde{w}_n(1) = 0$ and using (57), we obtain

$$\frac{1}{\lambda_n}\widetilde{w}_{n} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{58}$$

Estimates on $\widetilde{w}_n$ and $\lambda_n w_n$ Taking the inner product of $(49)_6$ with $\dfrac{i\widetilde{w}_n}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we see that

$$\rho_1 \left\|\widetilde{w}_n\right\|^2_{L^2(0,1)} + k_0 \left\langle (w_{nx} - l\varphi_n), \frac{i\widetilde{w}_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)}$$
$$+ lk \left\langle (\varphi_{nx} + \psi_n + l w_n), \frac{i\widetilde{w}_n}{\lambda_n}\right\rangle_{L^2(0,1)} + \delta \left\langle \frac{\theta_{nx}}{\lambda_n}, i\widetilde{w}_n\right\rangle_{L^2(0,1)} \longrightarrow 0.$$

Using (46), (47), (52), (57) and (58), we obtain

$$\widetilde{w}_n \longrightarrow 0 \quad \text{in } L^2(0,1), \tag{59}$$

and with $(49)_5$, we find

$$\lambda_n w_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{60}$$

Estimates on φ_{nx}, $\widetilde{\varphi}_n$ and $\lambda_n\varphi_n$ First, taking the inner product of $(\varphi_{nx} + \psi_n + l w_n)$ with $i\lambda_n\widetilde{w}_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we have

$$\begin{aligned}
\left\langle (\varphi_{nx} + \psi_n + l w_n), i\lambda_n \widetilde{w}_n\right\rangle_{L^2(0,1)} &= -\left\langle i\lambda_n\varphi_{nx}, \widetilde{w}_n\right\rangle_{L^2(0,1)} - \left\langle i\lambda_n\psi_n, \widetilde{w}_n\right\rangle_{L^2(0,1)} - l\left\langle i\lambda_n w_n, \widetilde{w}_n\right\rangle_{L^2(0,1)}\\
&= \left\langle \left(i\lambda_n\varphi_n - \widetilde{\varphi}_n\right), \widetilde{w}_{nx}\right\rangle_{L^2(0,1)} + \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx}\right\rangle_{L^2(0,1)} - \left\langle \left(i\lambda_n\psi_n - \widetilde{\psi}_n\right), \widetilde{w}_n\right\rangle_{L^2(0,1)}\\
&\quad - \left\langle \widetilde{\psi}_n, \widetilde{w}_n\right\rangle_{L^2(0,1)} - l\left\langle \left(i\lambda_n w_n - \widetilde{w}_n\right), \widetilde{w}_n\right\rangle_{L^2(0,1)} - l\left\|\widetilde{w}_n\right\|^2_{L^2(0,1)}\\
&= -\left\langle \left(i\lambda_n\varphi_{nx} - \widetilde{\varphi}_{nx}\right), \widetilde{w}_n\right\rangle_{L^2(0,1)} + \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx}\right\rangle_{L^2(0,1)} - \left\langle \left(i\lambda_n\psi_n - \widetilde{\psi}_n\right), \widetilde{w}_n\right\rangle_{L^2(0,1)}\\
&\quad - \left\langle \widetilde{\psi}_n, \widetilde{w}_n\right\rangle_{L^2(0,1)} - l\left\langle \left(i\lambda_n w_n - \widetilde{w}_n\right), \widetilde{w}_n\right\rangle_{L^2(0,1)} - l\left\|\widetilde{w}_n\right\|^2_{L^2(0,1)}.
\end{aligned}$$

Then, using (46), $(49)_1$, $(49)_3$, $(49)_5$ and (59), we deduce that

$$\left\langle (\varphi_{nx} + \psi_n + l w_n), i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} - \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{61}$$

Second, taking the inner product of $\widetilde{\varphi}_n$ with $\widetilde{w}_{nx}$ in $L^2(0,1)$, we arrive at

$$\begin{aligned}
\left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} &= \left\langle \widetilde{\varphi}_n, \left(\widetilde{w}_{nx} - l\widetilde{\varphi}_n\right) \right\rangle_{L^2(0,1)} + l \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} \\
&= -\left\langle \widetilde{\varphi}_n, \left(i\lambda_n w_{nx} - \widetilde{w}_{nx}\right) \right\rangle_{L^2(0,1)} + \left\langle \widetilde{\varphi}_n, l\left(i\lambda_n \varphi_n - \widetilde{\varphi}_n\right) \right\rangle_{L^2(0,1)} \\
&\quad + \left\langle \widetilde{\varphi}_n, i\lambda_n (w_{nx} - l\varphi_n) \right\rangle_{L^2(0,1)} + l \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)},
\end{aligned}$$

then, by (46), $(49)_1$ and $(49)_5$, we have

$$\lambda_n \left\langle \widetilde{\varphi}_n, i(w_{nx} - l\varphi_n) \right\rangle_{L^2(0,1)} - \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} + l \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{62}$$

Third, taking the inner product of $(49)_2$ with $(w_{nx} - l\varphi_n)$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we find

$$\begin{aligned}
&\left\langle i\lambda_n \rho_1 \widetilde{\varphi}_n, (w_{nx} - l\varphi_n) \right\rangle_{L^2(0,1)} + k \left\langle (\varphi_{nx} + \psi_n + l w_n), (w_{nx} - l\varphi_n)_x \right\rangle_{L^2(0,1)} \\
&-l k_0 \left\| (w_{nx} - l\varphi_n) \right\|^2_{L^2(0,1)} \longrightarrow 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\lambda_n \rho_1 \left\langle i\widetilde{\varphi}_n, (w_{nx} - l\varphi_n) \right\rangle_{L^2(0,1)} \\
&-\frac{k}{k_0} \left\langle (\varphi_{nx} + \psi_n + l w_n), \left[i\lambda_n \rho_1 \widetilde{w}_n - k_0 (w_{nx} - l\varphi_n)_x + l k (\varphi_{nx} + \psi_n + l w_n) + \delta \theta_{nx} \right] \right\rangle_{L^2(0,1)} \\
&+\frac{k\rho_1}{k_0} \left\langle (\varphi_{nx} + \psi_n + l w_n), i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} + \frac{l k^2}{k_0} \left\| (\varphi_{nx} + \psi_n + l w_n) \right\|^2_{L^2(0,1)} \\
&+\frac{\delta k}{k_0} \left\langle (\varphi_{nx} + \psi_n + l w_n), \theta_{nx} \right\rangle_{L^2(0,1)} - l k_0 \left\| (w_{nx} - l\varphi_n) \right\|^2_{L^2(0,1)} \longrightarrow 0.
\end{aligned}$$

Using (46), $(49)_6$, (52), (54) and (56) , we see that

$$\begin{aligned}
&-\lambda_n \rho_1 \left\langle \widetilde{\varphi}_n, i\ (w_{nx} - l\varphi_n) \right\rangle_{L^2(0,1)} + \frac{k\rho_1}{k_0} \left\langle (\varphi_{nx} + \psi_n + l w_n), i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} \\
&+\frac{l k^2}{k_0} \left\| (\varphi_{nx} + \psi_n + l w_n) \right\|^2_{L^2(0,1)} \longrightarrow 0.
\end{aligned} \tag{63}$$

Then, multiplying (61) by $\frac{-k\rho_1}{k_0}$ and (62) by ρ_1, and adding the obtained limits and (63), we obtain

$$\left(\frac{k}{k_0} - 1\right) \rho_1 \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} + \frac{l k^2}{k_0} \left\| (\varphi_{nx} + \psi_n + l w_n) \right\|^2_{L^2(0,1)} + \rho_1 l \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{64}$$

So, because $k = k_0$ (according to (25)), we get from (54) and (64) that

$$\varphi_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1) \tag{65}$$

and

$$\widetilde{\varphi}_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{66}$$

Moreover, $(49)_1$ and (66) give

$$\lambda_n \varphi_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{67}$$

Estimates on $\widetilde{\psi}_n$ and $\lambda_n \psi_n$ First, taking the inner product of (49)$_4$ with $(\varphi_{nx} + \psi_n + l w_n)$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}&\left\langle i\lambda_n \rho_2 \widetilde{\psi}_n, \varphi_{nx}\right\rangle_{L^2(0,1)} + \left\langle i\lambda_n \rho_2 \widetilde{\psi}_n, \psi_n\right\rangle_{L^2(0,1)} + l\left\langle i\lambda_n \rho_2 \widetilde{\psi}_n, w_n\right\rangle_{L^2(0,1)}\\ &\quad + b\left\langle \psi_{nx}, (\varphi_{nx} + \psi_n + l w_n)_x\right\rangle_{L^2(0,1)} + k\left\|(\varphi_{nx} + \psi_n + l w_n)\right\|^2_{L^2(0,1)} \longrightarrow 0,\end{aligned}$$

then

$$\begin{aligned}&-\lambda_n \rho_2 \left\langle \widetilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \rho_2\left\langle \widetilde{\psi}_n, \left(i\lambda_n \psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)}\\ &-l\rho_2\left\langle \widetilde{\psi}_n, \left(i\lambda_n w_n - \widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - l\rho_2\left\langle \widetilde{\psi}_n, \widetilde{w}_n\right\rangle_{L^2(0,1)}\\ &-\frac{b}{k}\left\langle \psi_{nx}, \left[i\lambda_n \rho_1 \widetilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l w_n)_x - l k_0 (w_{nx} - l\varphi_n)\right]\right\rangle_{L^2(0,1)}\\ &+\frac{b}{k}\left\langle \psi_{nx}, i\lambda_n \rho_1 \widetilde{\varphi}_n\right\rangle_{L^2(0,1)} - \frac{l k_0 b}{k}\left\langle \psi_{nx}, (w_{nx} - l\varphi_n)\right\rangle_{L^2(0,1)} + k\left\|\varphi_{nx} + \psi_n + l w_n\right\|^2_{L^2(0,1)} \longrightarrow 0,\end{aligned}$$

using (46), (49)$_2$, (49)$_3$, (49)$_5$, (54), (56), (59) and (65), we get

$$-\lambda_n \rho_2 \left\langle \widetilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} + \frac{b\rho_1}{k}\lambda_n\left\langle \psi_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{68}$$

Second, using the equality

$$\lambda_n\left\langle \psi_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} = -\left\langle \left(i\lambda_n \psi_{nx} - \widetilde{\psi}_{nx}\right), \widetilde{\varphi}_n\right\rangle_{L^2(0,1)} - \left\langle \widetilde{\psi}_{nx}, \widetilde{\varphi}_n\right\rangle_{L^2(0,1)},$$

integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned}\lambda_n\left\langle \psi_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} &= -\left\langle \left(i\lambda_n \psi_{nx} - \widetilde{\psi}_{nx}\right), \widetilde{\varphi}_n\right\rangle_{L^2(0,1)} + \left\langle \widetilde{\psi}_n, \widetilde{\varphi}_{nx}\right\rangle_{L^2(0,1)}\\ &= -\left\langle \left(i\lambda_n \psi_{nx} - \widetilde{\psi}_{nx}\right), \widetilde{\varphi}_n\right\rangle_{L^2(0,1)} - \left\langle \widetilde{\psi}_n, \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)} + \left\langle \widetilde{\psi}_n, i\lambda_n \varphi_{nx}\right\rangle_{L^2(0,1)}.\end{aligned}$$

Therefore, from (46), (49)$_1$ and (49)$_3$, we see that

$$\lambda_n\left\langle \psi_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} - \lambda_n\left\langle \widetilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} \longrightarrow 0, \tag{69}$$

so, multiplying (69) by $-\rho_2$ and inserting the obtained limit into (68), we obtain

$$\frac{\lambda_n}{k}(b\rho_1 - k\rho_2)\left\langle \psi_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{70}$$

Now, we use the fact that $b\rho_1 - k\rho_2 = 0$ (condition (25)), we get from (70) that

$$\widetilde{\psi}_n \longrightarrow 0 \quad \text{in } L^2(0,1), \tag{71}$$

and by (49)$_3$ and (71), we deduce that

$$\lambda_n \psi_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{72}$$

Estimate on ψ_{nx} and conclusion Taking the inner product of (49)$_4$ with ψ_n in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$-\rho_2 \left\langle \widetilde{\psi}_n, i\lambda_n \psi_n \right\rangle_{L^2(0,1)} + b \left\|\psi_{nx}\right\|^2_{L^2(0,1)} + k \left\langle (\varphi_{nx} + \psi_n + l w_n), \psi_n \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

and using (46), (54) and (72), we obtain

$$\psi_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{73}$$

A combination of (53), (54), (56), (59), (65), (66), (71) and (73) leads to (50), which is a contradiction with (46). Hence, in case (1), (25) implies (27).

3.2.2 Case of system (4)

In case (4), the limit (48) implies the following ones:

$$\begin{cases} i\lambda_n \varphi_n - \widetilde{\varphi}_n \longrightarrow 0 \text{ in } H^1_*(0,1), \\ i\lambda_n \rho_1 \widetilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l w_n)_x - l k_0 (w_{nx} - l\varphi_n) \longrightarrow 0 \quad \text{in } L^2(0,1), \\ i\lambda_n \psi_n - \widetilde{\psi}_n \longrightarrow 0 \quad \text{in } H^1_*(0,1), \\ i\lambda_n \rho_2 \widetilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l w_n) \longrightarrow 0 \quad \text{in } L^2(0,1), \\ i\lambda_n w_n - \widetilde{w}_n \longrightarrow 0 \quad \text{in } H^1_*(0,1), \\ i\lambda_n \rho_1 \widetilde{w}_n - k_0(w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + l w_n) + \delta \widetilde{\theta}_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1), \\ i\lambda_n \theta_n - \widetilde{\theta}_n \longrightarrow 0 \quad \text{in } H^1_*(0,1), \\ i\lambda_n \rho_3 \widetilde{\theta}_n - \left(\beta\theta_n + \gamma \widetilde{\theta}_n\right)_{xx} + \delta \widetilde{w}_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \end{cases} \tag{74}$$

Estimates on $\lambda_n\theta_n$, $\lambda_n\theta_{nx}$, $\widetilde{\theta}_n$ and $\widetilde{\theta}_{nx}$ Taking the inner product of $(i\,\lambda_n\, I - \mathcal{A})\,\Phi_n$ with Φ_n in $\mathcal{H}$ and using (17), we find

$$Re\,\langle (i\,\lambda_n\, I - \mathcal{A})\,\Phi_n, \Phi_n\rangle_{\mathcal{H}} = \gamma \left\|\widetilde{\theta}_{nx}\right\|^2_{L^2(0,1)}. \tag{75}$$

Using (46) and (48), we deduce that

$$\widetilde{\theta}_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{76}$$

Because $\widetilde{\theta}_n(0) = 0$ and according to Poincaré's inequality, then we get from (76) that

$$\widetilde{\theta}_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{77}$$

The above two limits combined with (74)$_7$ give

$$\lambda_n \theta_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1) \tag{78}$$

and

$$\lambda_n \theta_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{79}$$

Estimates on φ_n, ψ_n and w_n Multiplying (74)$_1$, (74)$_3$ and (74)$_5$ by $\frac{1}{\lambda_n}$, and using (46) and (47), we find (54).

Estimate on $\frac{1}{\lambda_n} w_{nxx}$ As in case (1) (Sect. 3.2.1), applying triangle inequality and using (74)$_6$ and (76), we obtain (55).

Estimates on w_{nx}, $\frac{1}{\lambda_n}\widetilde{w}_{nx}$ and $\frac{1}{\lambda_n}\widetilde{w}_n$ Taking the inner product of (74)$_8$ with $\dfrac{i w_{nx}}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using (46), (47) and the boundary conditions, we get

$$\rho_3\left\langle\widetilde{\theta}_n, w_{nx}\right\rangle_{L^2(0,1)}+\beta\left\langle\theta_{nx}, \frac{i w_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)}+\gamma\left\langle\widetilde{\theta}_{nx}, \frac{i w_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)}$$
$$-\delta\left\langle\left(i\lambda_n w_{nx}-\widetilde{w}_{nx}\right), \frac{i w_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)}+\delta\,\|w_{nx}\|^2_{L^2(0,1)}\longrightarrow 0.$$

Using (55), (74)$_5$, (76), (77) and (78), we get (56). By multiplying (74)$_5$ by $\frac{1}{\lambda_n}$, we get (57). Moreover, because $\widetilde{w}_n(1)=0$, we have (58).

Estimates on $\widetilde{w}_n$ and $\lambda_n w_n$ As in case (1) (Sect. 3.2.1), taking the inner product of (74)$_6$ with $\dfrac{i\widetilde{w}_n}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we find (59) and (60).

Estimate on $\widetilde{\varphi}_n$ and conclusion The same computations as in case (1) (Sect. 3.2.1) imply (64) and (70), so (25) leads to (65), (66), (71) and (73). Consequently, (50) holds, which is a contradiction with (46). Hence, also in case (4), (25) implies (27).

3.3 Condition (27) implies (25)

We prove this implication by contradiction. So, we assume that (25) does not hold and prove that (27) is not satisfied; that is we prove that there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that

$$\lim_{n\to\infty}\left\|(i\lambda_n I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}=\infty,$$

which is equivalent to prove that there exists a sequence $(F_n)_n \subset \mathcal{H}$ satisfying

$$\|F_n\|_{\mathcal{H}}\le 1,\quad \forall n\in\mathbb{N} \tag{80}$$

and

$$\lim_{n\to\infty}\left\|(i\lambda_n I-\mathcal{A})^{-1}F_n\right\|_{\mathcal{H}}=\infty. \tag{81}$$

For this purpose, let

$$\Phi_n=(i\lambda_n I-\mathcal{A})^{-1}F_n,\quad \forall n\in\mathbb{N}.$$

Then we have to prove that (80) holds such that

$$\lim_{n\to\infty}\|\Phi_n\|_{\mathcal{H}}=\infty\quad\text{and}\quad i\lambda_n\Phi_n-\mathcal{A}\Phi_n=F_n,\ \forall n\in\mathbb{N}. \tag{82}$$

Taking

$$\Phi_n=\begin{cases}\left(\varphi_n,\tilde{\varphi}_n,\psi_n,\tilde{\psi}_n,w_n,\tilde{w}_n,\theta_n\right)^T & \text{in case (1)},\\ \left(\varphi_n,\tilde{\varphi}_n,\psi_n,\tilde{\psi}_n,w_n,\tilde{w}_n,\theta_n,\tilde{\theta}_n\right)^T & \text{in case (4)}\end{cases}$$

and

$$F_n=\begin{cases}(f_{1n},\ldots,f_{7n})^T & \text{in case (1)},\\ (f_{1n},\ldots,f_{8n})^T & \text{in case (4)}.\end{cases}$$

Then, from the second equality in (82), we have the following systems:

$$\begin{cases} i\lambda_n\varphi_n - \tilde{\varphi}_n = f_{1n}, \\ i\rho_1\lambda_n\tilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) = \rho_1 f_{2n}, \\ i\lambda_n\psi_n - \tilde{\psi}_n \;= f_{3n}, \\ i\rho_2\lambda_n\tilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\ i\lambda_n w_n - \tilde{w}_n = f_{5n}, \\ i\rho_1\lambda_n\tilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) + \delta\theta_{nx} = \rho_1 f_{6n}, \\ i\rho_3\lambda_n\theta_n - \beta\theta_{nxx} + \delta\tilde{w}_{nx} = \rho_3 f_{7n} \end{cases} \tag{83}$$

in case (1), and

$$\begin{cases} i\lambda_n\varphi_n - \tilde{\varphi}_n = f_{1n}, \\ i\rho_1\lambda_n\tilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) = \rho_1 f_{2n}, \\ i\lambda_n\psi_n - \tilde{\psi}_n \;= f_{3n}, \\ i\rho_2\lambda_n\tilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\ i\lambda_n w_n - \tilde{w}_n = f_{5n}, \\ i\rho_1\lambda_n\tilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) + \delta\tilde{\theta}_{nx} = \rho_1 f_{6n}, \\ i\lambda_n\theta_n - \tilde{\theta}_n = f_{7n}, \\ i\rho_3\lambda_n\tilde{\theta}_n - \left(\beta\theta_n + \gamma\tilde{\theta}_n\right)_{xx} + \delta\tilde{w}_{nx} = \rho_3 f_{8n} \end{cases} \tag{84}$$

in case (4). Choosing

$$f_{4n}(x) = c\cos(Nx), \quad f_{1n} = f_{2n} = f_{3n} = f_{5n} = f_{6n}(x) = f_{7n} = f_{8n} = 0, \tag{85}$$

where $N = \frac{(2n+1)\pi}{2}$ and c is a constant satisfying $0 < |c| \leq \frac{1}{\sqrt{\rho_2}}$, so

$$\|F_n\|^2_{\mathcal{H}} = \rho_2\,\|f_{4n}\|^2_{L^2(0,1)} = \rho_2|c|^2\int_0^1 \cos^2(Nx)\,\mathrm{d}x \leq 1.$$

On the other hand, the systems (83) and (84) become, respectively,

$$\begin{cases} \tilde{\varphi}_n = i\lambda_n\varphi_n, \quad \tilde{\psi}_n = i\lambda_n\psi_n, \quad \tilde{w}_n = i\lambda_n w_n, \\ -\rho_1\lambda_n^2\varphi_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) = 0, \\ -\rho_2\lambda_n^2\psi_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\ -\rho_1\lambda_n^2 w_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) + \delta\theta_{nx} = 0, \\ i\rho_3\lambda_n\theta_n - \beta\theta_{nxx} + i\delta\lambda_n w_{nx} = 0 \end{cases} \tag{86}$$

and

$$\begin{cases} \tilde{\varphi}_n = i\lambda_n\varphi_n, \quad \tilde{\psi}_n = i\lambda_n\psi_n, \quad \tilde{w}_n = i\lambda_n w_n, \quad \tilde{\theta}_n = i\lambda_n\theta_n, \\ -\rho_1\lambda_n^2\varphi_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) = 0, \\ -\rho_2\lambda_n^2\psi_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\ -\rho_1\lambda_n^2 w_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) + i\delta\lambda_n\theta_{nx} = 0, \\ -i\rho_3\lambda_n^2\theta_n - \left(\beta\theta_n + i\gamma\lambda_n\theta_n\right)_{xx} + i\delta\lambda_n w_{nx} = 0. \end{cases} \tag{87}$$

Let us consider the choices

$$\begin{cases} \varphi_n(x) = \alpha_1 \sin\,(Nx)\,, \;\; \psi_n(x) = \alpha_2 \cos\,(Nx)\,, \;\; w_n(x) = \alpha_3 \cos\,(Nx)\,, \\ \theta_n(x) = \alpha_4 \sin\,(Nx)\,, \;\; \tilde{\varphi}_n(x) = i\lambda_n \alpha_1 \sin\,(Nx)\,, \;\; \tilde{\psi}_n(x) = i\lambda_n \alpha_2 \cos\,(Nx)\,, \\ \tilde{w}_n(x) = i\lambda_n \alpha_3 \cos\,(Nx)\,, \;\; \tilde{\theta}_n(x) = i\lambda_n \alpha_4 \sin\,(Nx)\,, \end{cases}$$

where $\alpha_1, \ldots, \alpha_4$ are constants depending on N (will be fixed later). Then the last equation in (86) and the last one in (87) are equivalent to $\alpha_4 = \mu_n N \alpha_3$, where

$$\mu_n = \begin{cases} \frac{i\delta\lambda_n}{\beta N^2 + i\rho_3\lambda_n} & \text{in case (86)}, \\ \frac{i\delta\lambda_n}{i\gamma\lambda_n N^2 + \beta N^2 - i\rho_3\lambda_n^2} & \text{in case (87)}. \end{cases} \tag{88}$$

Therefore, (86) and (87) are satisfied if and only if

$$\begin{cases} \left[kN^2 + l^2 k_0 - \rho_1 \lambda_n^2\right] \alpha_1 + kN\alpha_2 + l\,(k + k_0)\,N\alpha_3 = 0, \\ \left[bN^2 + k - \rho_2 \lambda_n^2\right] \alpha_2 + kN\alpha_1 + lk\alpha_3 = \rho_2 c, \\ \left[(k_0 + \delta_n \mu_n)\,N^2 + l^2 k - \rho_1 \lambda_n^2\right] \alpha_3 + l\,(k + k_0)\,N\alpha_1 + lk\alpha_2 = 0, \end{cases} \tag{89}$$

where

$$\delta_n = \begin{cases} \delta & \text{in case (86)}, \\ i\delta\lambda_n & \text{in case (87)}. \end{cases}$$

Because (25) is assumed to be not satisfied, then

$$\rho_1 b - \rho_2 k \neq 0 \quad \text{or} \quad [\rho_1 b - \rho_2 k = 0 \text{ and } k - k_0 \neq 0],$$

so we distinguish these two cases.

Case 1 $\rho_1 b - \rho_2 k \neq 0$. Let choose $\lambda_n = \sqrt{\dfrac{b}{\rho_2} N^2 + \dfrac{kk_0}{\rho_2\,(k + k_0)}}$, then

$$\lim_{n\to\infty} \delta_n \mu_n = 0 \quad \text{and} \quad N^2 \delta_n \mu_n \sim \begin{cases} \frac{i\delta^2}{\beta} \lambda_n & \text{in case (86)}, \\ \frac{i\delta^2}{\gamma} \lambda_n & \text{in case (87)}. \end{cases} \tag{90}$$

On the other hand, (89) becomes

$$\begin{cases} \left[\left(k - \dfrac{\rho_1 b}{\rho_2}\right) N^2 + l^2 k_0 - \dfrac{\rho_1 k k_0}{\rho_2\,(k + k_0)}\right] \alpha_1 + kN\alpha_2 + l\,(k + k_0)\,N\alpha_3 = 0, \\ \dfrac{k^2}{k + k_0} \alpha_2 + kN\alpha_1 + lk\alpha_3 = \rho_2 c, \\ \left[\left(k_0 - \dfrac{\rho_1 b}{\rho_2} + \delta_n \mu_n\right) N^2 + l^2 k - \dfrac{\rho_1 k k_0}{\rho_2\,(k + k_0)}\right] \alpha_3 + l\,(k + k_0)\,N\alpha_1 + lk\alpha_2 = 0. \end{cases} \tag{91}$$

From $(91)_2$ we get

$$\alpha_1 = \frac{\rho_2 c - lk\alpha_3 - \dfrac{k^2}{k + k_0} \alpha_2}{kN}. \tag{92}$$

By substituting (92) into $(91)_3$ and into $(91)_1$, we obtain, respectively,

$$\alpha_3 = \frac{\rho_2 l c (k + k_0)}{k\left[\left(\dfrac{\rho_1 b}{\rho_2} - k_0 - \delta_n \mu_n\right) N^2 + l^2 k_0 + \dfrac{\rho_1 k k_0}{\rho_2\,(k + k_0)}\right]} \tag{93}$$

and

$$\alpha_2 = \frac{\left[(\rho_2 c - lk\alpha_3)\left(k - \frac{\rho_1 b}{\rho_2}\right) + lk(k+k_0)\alpha_3\right]N^2 + (\rho_2 c - lk\alpha_3)\left[l^2 k_0 - \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right]}{\frac{k^2}{k+k_0}\left[-\left(\frac{\rho_1 b}{\rho_2} + k_0\right)N^2 + l^2 k_0 - \frac{\rho_1 k k_0}{\rho_2 (k+k_0)}\right]}. \tag{94}$$

According to (90), we see that (93) implies that

$$\lim_{n\to\infty} \alpha_3 = 0;$$

therefore,

$$\lim_{n\to\infty} \alpha_2 = \frac{c(k+k_0)(\rho_1 b - \rho_2 k)}{k^2\left(\frac{\rho_1 b}{\rho_2} + k_0\right)} \neq 0$$

since $\rho_1 b - \rho_2 k \neq 0$. Then

$$\lim_{n\to\infty} |\alpha_2| N = \infty. \tag{95}$$

Finally, using the norm of ψ_{nx} in $L^2(0,1)$, we obtain

$$\begin{aligned} \|\Phi_n\|^2_{\mathcal{H}} &\geq b\,\|\psi_{nx}\|^2_{L^2(0,1)} = b|\alpha_2|^2 N^2 \int_0^1 \sin^2(Nx)\,\mathrm{d}x \\ &\geq \frac{b}{2}|\alpha_2|^2 N^2 \int_0^1 (1-\cos(2Nx))\,\mathrm{d}x = \frac{b}{2}|\alpha_2|^2 N^2 \longrightarrow \infty. \end{aligned} \tag{96}$$

Case 2 $\rho_1 b - \rho_2 k = 0$ and $k - k_0 \neq 0$. Let choose $\lambda_n = \sqrt{\frac{k}{\rho_1}N^2 + \frac{k}{\sqrt{\rho_1\rho_2}}N}$. Then (89) becomes

$$\begin{cases} \left(-\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}}N + l^2 k_0\right)\alpha_1 + kN\alpha_2 + l(k+k_0)N\alpha_3 = 0, \\ \left(-\frac{\rho_2 k}{\sqrt{\rho_1\rho_2}}N + k\right)\alpha_2 + kN\alpha_1 + lk\alpha_3 = \rho_2 c, \\ \left[(k_0 - k + \delta_n\mu_n)N^2 - \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}}N + l^2 k\right]\alpha_3 + l(k+k_0)N\alpha_1 + lk\alpha_2 = 0. \end{cases} \tag{97}$$

From $(97)_1$ we get, for $N > \frac{l^2 k_0\sqrt{\rho_1\rho_2}}{\rho_1 k}$,

$$\alpha_1 = \frac{kN\alpha_2 + l(k+k_0)N\alpha_3}{\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}}N - l^2 k_0}. \tag{98}$$

By substituting (98) into $(97)_3$, we find, for $N > \frac{l^2 k_0\sqrt{\rho_1\rho_2}}{\rho_1 k}$,

$$\alpha_3 = \frac{lk\left[(k+k_0)N^2 + \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}}N - l^2 k_0\right]\alpha_2}{\left(-\frac{\rho_1 k}{\sqrt{\rho_1\rho_2}}N + l^2 k_0\right)\left[(k_0 - k + \delta_n\mu_n)N^2 - \frac{\rho_1 k}{\sqrt{\rho_1\rho_2}}N + l^2 k\right] - l^2(k+k_0)^2 N^2}. \tag{99}$$

By substituting (98) and (99) into $(97)_2$, we obtain, for $N > \frac{l^2 k_0\sqrt{\rho_1\rho_2}}{\rho_1 k}$,

$$\alpha_2 = \frac{a_1}{a_2}, \tag{100}$$

where

$$a_1 = -\rho_2 c \left(\frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N - l^2 k_0 \right)^2 \left[(k_0 - k + \delta_n \mu_n) N^2 - \frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N + l^2 k \right]$$
$$+ \rho_2 c l^2 (k + k_0)^2 \left(-\frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N + l^2 k_0 \right) N^2$$

and

$$a_2 = l^2 k^2 \left[(k + k_0) N^2 + \frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N - l^2 k_0 \right]^2 + l^2 (k + k_0)^2 \left(l^2 k k_0 - \frac{\rho_1 k^2 + l^2 k k_0 \rho_2}{\sqrt{\rho_1 \rho_2}} N \right) N^2$$
$$+ \left(l^2 k k_0 - \frac{\rho_1 k^2 + l^2 k k_0 \rho_2}{\sqrt{\rho_1 \rho_2}} N \right) \left(\frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N - l^2 k_0 \right) \left[(k_0 - k + \delta_n \mu_n) N^2 - \frac{\rho_1 k}{\sqrt{\rho_1 \rho_2}} N + l^2 k \right].$$

We see that (90) and (100) imply that

$$\lim_{n \to \infty} |\alpha_2| = \begin{cases} \left| \frac{c \rho_1 \rho_2 (k - k_0)}{k [\rho_2 l^2 (k + 3k_0) + \rho_1 (k - k_0)]} \right| & \text{if } \rho_2 l^2 (k + 3k_0) + \rho_1 (k - k_0) \neq 0, \\ \infty & \text{if } \rho_2 l^2 (k + 3k_0) + \rho_1 (k - k_0) = 0. \end{cases} \tag{101}$$

Because $k - k_0 \neq 0$, then (95) holds. Consequently, (96) remains valid.

Finally, the equivalence between (27) and (25) is established, and consequently, the proof of Theorem 3.1 is completed. □

4 Polynomial stability

In this section, we prove the following polynomial stability independently from (25):

Theorem 4.1 *Assume that (15) and (24) hold. Then, for any $m \in \mathbb{N}^*$, there exists a constant $c_m > 0$ such that, for any $\Phi_0 \in D(\mathcal{A}^m)$ and $t > 0$,*

$$\left\| e^{t\mathcal{A}} \Phi_0 \right\|_{\mathcal{H}} \leq \begin{cases} c_m \|\Phi_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t} \right)^{\frac{m}{4}} \ln t & \text{if } \rho_1 b - \rho_2 k = 0, \\ c_m \|\Phi_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t} \right)^{\frac{m}{10}} \ln t & \text{if } \rho_1 b - \rho_2 k \neq 0. \end{cases} \tag{102}$$

The key of the proof of Theorem 4.1 is the following known theorem:

Theorem 4.2 [12] *If a bounded C_0 semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ satisfies (26) and, for some $j \in \mathbb{N}^*$,*

$$\sup_{|\lambda| \geq 1} \frac{1}{\lambda^j} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{103}$$

Then, for any $m \in \mathbb{N}^*$, there exists a positive constant c_m such that

$$\left\| e^{t\mathcal{A}} z_0 \right\|_{\mathcal{H}} \leq c_m \|z_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t} \right)^{\frac{m}{j}} \ln t, \quad \forall z_0 \in D(\mathcal{A}^m), \ \forall t > 0. \tag{104}$$

Proof In Sect. 3, we have proved that (24) implies (26). Then we only need to show (103), where $j = 4$ if $\rho_1 b - \rho_2 k = 0$, and $j = 10$ if $\rho_1 b - \rho_2 k \neq 0$. Let us establish (103) by contradiction. Assume that (103) is false, then there exist sequences $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying (46), (47) and

$$\lim_{n \to \infty} \lambda_n^j \|(i\lambda_n I - \mathcal{A}) \Phi_n\|_{\mathcal{H}} = 0. \tag{105}$$

To get a contradiction with (46), we use similar arguments to the ones used in Sect. 3.2. Let

$$\Phi_n = \begin{cases} \left(\varphi_n, \widetilde{\varphi}_n, \psi_n, \widetilde{\psi}_n, w_n, \widetilde{w}_n, \theta_n \right)^T & \text{in case (1)} \\ \left(\varphi_n, \widetilde{\varphi}_n, \psi_n, \widetilde{\psi}_n, w_n, \widetilde{w}_n, \theta_n, \widetilde{\theta}_n \right)^T & \text{in case (4).} \end{cases}$$

4.1 Case of system (1) with $\rho_1 b - \rho_2 k = 0$

The limit (105) with $j = 4$ implies that

$$\begin{cases} \lambda_n^4 \left[i\lambda_n \varphi_n - \widetilde{\varphi}_n \right] \to 0 \quad \text{in } H_*^1(0,1), \\ \lambda_n^4 \left[i\rho_1 \lambda_n \widetilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l w_n)_x - l k_0 (w_{nx} - l\varphi_n) \right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n \psi_n - \widetilde{\psi}_n \right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_2 \lambda_n \widetilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l w_n) \right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n w_n - \widetilde{w}_n \right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_1 \lambda_n \widetilde{w}_n - k_0 (w_{nx} - l\varphi_n)_x + l k (\varphi_{nx} + \psi_n + l w_n) + \delta\theta_{nx} \right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^4 \left[i\rho_3 \lambda_n \theta_n - \beta\theta_{nxx} + \delta\widetilde{w}_{nx} \right] \to 0 \quad \text{in } L^2(0,1). \end{cases} \tag{106}$$

Estimates on θ_{nx} and θ_n Taking the inner product of $\lambda_n^4 (i\lambda_n I - \mathcal{A})\ \Phi_n$ with Φ_n in $\mathcal{H}$ and using (17), we get

$$Re\left\langle \lambda_n^4 (i\lambda_n I - \mathcal{A})\ \Phi_n, \Phi_n \right\rangle_{\mathcal{H}} = Re\left(i\lambda_n^5 \|\Phi_n\|^2_{L^2(0,1)} + \beta\lambda_n^4 \|\theta_{nx}\|^2_{L^2(0,1)} \right) = \beta\lambda_n^4 \|\theta_{nx}\|^2_{L^2(0,1)}.$$

So (46) and (105) imply that

$$\lambda_n^2 \theta_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{107}$$

Because θ_n in $H_*^1(0,1)$ and thanks to Poincaré's inequality, we deduce that

$$\lambda_n^2 \theta_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{108}$$

Estimates on φ_n, ψ_n and w_n Multiplying $(106)_1$, $(106)_3$ and $(106)_5$ by $\frac{1}{\lambda_n^5}$, and using (46) and (47), we obtain (54).

Estimate on $\frac{1}{\lambda_n} w_{nxx}$ Multiplying $(106)_6$ by $\frac{1}{\lambda_n^5}$ and using (46), (47) and (107), we conclude (55).

Estimates on $\lambda_n w_{nx}$, $\lambda_n w_n$, $\widetilde{w}_{nx}$ and $\widetilde{w}_n$ Taking the inner product of $(106)_7$ with $\frac{i}{\lambda_n^3} w_{nx}$ in $L^2(0,1)$ and using (46) and (47), we get

$$\rho_3 \left\langle \lambda_n^2 \theta_n, w_{nx} \right\rangle_{L^2(0,1)} - \beta \left\langle \lambda_n \theta_{nxx}, i w_{nx} \right\rangle_{L^2(0,1)}$$
$$-\delta \left\langle \lambda_n \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right), i w_{nx} \right\rangle_{L^2(0,1)} + \delta\lambda_n^2 \|w_{nx}\|^2_{L^2(0,1)} \longrightarrow 0,$$

then, integrating by parts and using the boundary conditions, we deduce that

$$\rho_3 \left\langle \lambda_n^2 \theta_n, w_{nx} \right\rangle_{L^2(0,1)} + \beta \left\langle \lambda_n^2 \theta_{nx}, \frac{i}{\lambda_n} w_{nxx} \right\rangle_{L^2(0,1)}$$
$$-\delta \left\langle \lambda_n \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right), i w_{nx} \right\rangle_{L^2(0,1)} + \delta\lambda_n^2 \|w_{nx}\|^2_{L^2(0,1)} \longrightarrow 0. \tag{109}$$

Combining (46), (47), (55), $(106)_5$, (107) and (108), we get

$$\lambda_n w_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{110}$$

Moreover, again by multiplying $(106)_5$ by $\frac{1}{\lambda_n^4}$, we find

$$\widetilde{w}_{nx} \to 0 \quad \text{in } L^2(0,1)\,, \tag{111}$$

and, as w_n, $\widetilde{w}_n \in \tilde{H}{}^1_*(0,1)$ and thanks to Poincaré's inequality, we have also (59) and (60).

Estimates on $\lambda_n^2 w_n$ and $\lambda_n \widetilde{w}_n$ Multiplying $(106)_1$ and $(106)_3$ by $\dfrac{1}{\lambda_n^4}$, and using (46) and (47), we have

$$(\lambda_n \varphi_n)_n \text{ and } (\lambda_n \psi_n)_n \text{ are bounded in } L^2(0,1)\,. \tag{112}$$

Taking the inner product of $(106)_6$ with $\dfrac{i}{\lambda_n^3}\widetilde{w}_n$ in $L^2(0,1)$, integrating by parts and using (46), (47) and the boundary conditions, we get

$$\begin{aligned}&\rho_1 \left\|\lambda_n \widetilde{w}_n\right\|^2_{L^2(0,1)} + k_0 \left\langle \lambda_n (w_{nx} - l\varphi_n), i\widetilde{w}_{nx}\right\rangle_{L^2(0,1)} \\ &+lk\left\langle \lambda_n(\varphi_{nx} + \psi_n + lw_n), i\widetilde{w}_n\right\rangle_{L^2(0,1)} + \delta\left\langle \lambda_n\theta_{nx}, i\widetilde{w}_n\right\rangle_{L^2(0,1)} \to 0.\end{aligned} \tag{113}$$

So, using (59), (60), (107), (110), (111) and (112), we deduce that

$$\lambda_n \widetilde{w}_n \longrightarrow 0 \quad \text{in } L^2(0,1)\,, \tag{114}$$

and by multiplying $(106)_5$ by $\dfrac{1}{\lambda_n^3}$ and using (47), we find

$$\lambda_n^2 w_n \longrightarrow 0 \quad \text{in } L^2(0,1)\,. \tag{115}$$

Estimate on φ_{nx} Multiplying $(106)_2$ and $(106)_4$ by $\dfrac{1}{\lambda_n^5}$ and using (46) and (47), we get

$$\left(\frac{1}{\lambda_n}\varphi_{nxx}\right)_n \text{ and } \left(\frac{1}{\lambda_n}\psi_{nxx}\right)_n \text{ are bounded in } L^2(0,1)\,. \tag{116}$$

On the other hand, taking the inner product of $(106)_6$ with $\dfrac{1}{\lambda_n^4}\varphi_{nx}$ in $L^2(0,1)$, integrating by parts and using (46), (47) and the boundary conditions, we get

$$\begin{aligned}&i\rho_1\left\langle \lambda_n \widetilde{w}_n, \varphi_{nx}\right\rangle_{L^2(0,1)} + \langle lk(\psi_n + lw_n) + \delta\theta_{nx}, \varphi_{nx}\rangle_{L^2(0,1)} \\ &+l(k+k_0)\left\|\varphi_{nx}\right\|^2_{L^2(0,1)} + k_0\left\langle \lambda_n w_{nx}, \frac{1}{\lambda_n}\varphi_{nxx}\right\rangle_{L^2(0,1)} \to 0.\end{aligned} \tag{117}$$

Then, using (54), (107), (110), (114) and (116), we deduce that

$$\varphi_{nx} \to 0 \text{ in } L^2(0,1)\,. \tag{118}$$

Estimates on $\lambda_n \varphi_n$ and $\widetilde{\varphi}_n$ Taking the inner product of $(106)_2$ with $\dfrac{1}{\lambda_n^4}\varphi_n$ in $L^2(0,1)$, using (46) and (47), integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned}&-\rho_1\left\langle \widetilde{\varphi}_n, \left(i\lambda_n\varphi_n - \widetilde{\varphi}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\left\|\widetilde{\varphi}_n\right\|^2_{L^2(0,1)} \\ &+k\left\langle(\varphi_{nx} + \psi_n + lw_n), \varphi_{nx}\right\rangle_{L^2(0,1)} - lk_0\left\langle(w_{nx} - l\varphi_n), \varphi_n\right\rangle_{L^2(0,1)} \to 0,\end{aligned}$$

then, using (54), $(106)_1$ and (118), we find

$$\widetilde{\varphi}_n \to 0 \quad \text{in } L^2(0,1)\,. \tag{119}$$

Moreover, multiplying $(106)_1$ by $\dfrac{1}{\lambda_n^4}$ and using (47) and (119), we get

$$\lambda_n \varphi_n \to 0 \quad \text{in } L^2(0,1)\,. \tag{120}$$

Estimates on ψ_{nx} and $\widetilde{\psi}_n$ and conclusion First, taking the inner product of $(106)_4$ with $\dfrac{1}{\lambda_n^4}\psi_n$ in $L^2(0,1)$, using (46) and (47), integrating by parts and using the boundary conditions, we obtain

$$-\rho_2 \left\langle \widetilde{\psi}_n, \left(i\lambda_n \psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2 \left\| \widetilde{\psi}_n \right\|_{L^2(0,1)}^2$$
$$+ b \left\| \psi_{nx} \right\|_{L^2(0,1)}^2 + k \left\langle (\varphi_{nx} + \psi_n + l w_n), \psi_n \right\rangle_{L^2(0,1)} \to 0,$$

then, using (54) and $(106)_3$, we find

$$b \left\| \psi_{nx} \right\|_{L^2(0,1)}^2 - \rho_2 \left\| \widetilde{\psi}_n \right\|_{L^2(0,1)}^2 \to 0. \tag{121}$$

Second, taking the inner product of $(106)_2$ with $\dfrac{1}{\lambda_n^4}\psi_{nx}$, integrating by parts and using the boundary conditions, (46) and (47), we obtain

$$-k \left\| \psi_{nx} \right\|_{L^2(0,1)}^2 + k \left\langle \varphi_{nx}, \psi_{nxx} \right\rangle_{L^2(0,1)} + i\rho_1 \lambda_n \left\langle \widetilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)}$$
$$-l(k + k_0) \left\langle w_{nx}, \psi_{nx} \right\rangle_{L^2(0,1)} - l^2 k_0 \left\langle \varphi_{nx}, \psi_n \right\rangle_{L^2(0,1)} \to 0 \quad \text{in } L^2(0,1)\,.$$

Exploiting (110) and (118), we get

$$-k \left\| \psi_{nx} \right\|_{L^2(0,1)}^2 + k \left\langle \varphi_{nx}, \psi_{nxx} \right\rangle_{L^2(0,1)} + i\rho_1 \lambda_n \left\langle \widetilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} \to 0 \quad \text{in } L^2(0,1)\,. \tag{122}$$

Third, taking the inner product of $\dfrac{k}{b\lambda_n^4}\varphi_{nx}$ with $(106)_4$, integrating by parts and using the boundary conditions, (46) and (47), we obtain

$$-k \left\langle \varphi_{nx}, \psi_{nxx} \right\rangle_{L^2(0,1)} - \frac{i\rho_2 k \lambda_n}{b} \left\langle \widetilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} + \frac{k^2}{b} \left\langle \varphi_{nx}, (\varphi_{nx} + \psi_n + l w_n) \right\rangle_{L^2(0,1)}$$
$$+ \frac{k\rho_2}{b} \left\langle \varphi_n, i\lambda_n \left(i\lambda_n \psi_{nx} - \widetilde{\psi}_{nx}\right)\right\rangle_{L^2(0,1)} - \frac{ik\rho_2}{b} \left\langle \lambda_n \left(i\lambda_n \varphi_n - \widetilde{\varphi}_n\right), \psi_{nx} \right\rangle_{L^2(0,1)} \to 0 \quad \text{in } L^2(0,1)\,,$$

so, from $(106)_1$, $(106)_3$ and (118), we find

$$-k \left\langle \varphi_{nx}, \psi_{nxx} \right\rangle_{L^2(0,1)} - \frac{i\rho_2 k \lambda_n}{b} \left\langle \widetilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} \to 0 \quad \text{in } L^2(0,1)\,. \tag{123}$$

By adding (122) and (123) and using the equality $\rho_1 b - \rho_2 k = 0$, we see that

$$\psi_{nx} \to 0 \quad \text{in } L^2(0,1)\,. \tag{124}$$

Therefore, from (121), we get

$$\widetilde{\psi}_n \to 0 \quad \text{in } L^2(0,1)\,. \tag{125}$$

Finally, the limits (54), (59), (108), (110), (118), (119), (124) and (125) imply (50), which is a contradiction with (46). Consequently, (103) with $j = 4$ holds.

4.2 Case of system (1) with $\rho_1 b - \rho_2 k \neq 0$

The limit (105) with $j = 10$ implies (106) with λ_n^{10} instead of λ_n^4; that is

$$\begin{cases} \lambda_n^{10}\left[i\lambda_n\varphi_n - \widetilde{\varphi}_n\right] \to 0 \quad \text{in } H_*^1(0,1), \\ \lambda_n^{10}\left[i\rho_1\lambda_n\widetilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + lw_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right)\right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^{10}\left[i\lambda_n\psi_n - \widetilde{\psi}_n\right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^{10}\left[i\rho_2\lambda_n\widetilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + lw_n\right)\right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^{10}\left[i\lambda_n w_n - \widetilde{w}_n\right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^{10}\left[i\rho_1\lambda_n\widetilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + lw_n\right) + \delta\theta_{nx}\right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^{10}\left[i\rho_3\lambda_n\theta_n - \beta\theta_{nxx} + \delta\widetilde{w}_{nx}\right] \to 0 \quad \text{in } L^2(0,1). \end{cases} \tag{126}$$

Similarly to the case $\rho_1 b - \rho_2 k = 0$ (Sect. 4.1), we see that (54), (55), (112), (116) and (118) hold (for (112) and (118), we have just to use $\frac{1}{\lambda_n^{10}}$ instead of $\frac{1}{\lambda_n^4}$, and for (116), we use $\frac{1}{\lambda_n^{11}}$ instead of $\frac{1}{\lambda_n^5}$).

Moreover, the same computations as in Sect. 4.1 (case $\rho_1 b - \rho_2 k = 0$) give (instead of (107), (108), (110), (111), (59) and (60))

$$\lambda_n^5\theta_{nx},\ \lambda_n^5\theta_n,\ |\lambda_n|^{\frac{5}{2}}w_{nx},\ |\lambda_n|^{\frac{3}{2}}\widetilde{w}_{nx},\ |\lambda_n|^{\frac{3}{2}}\widetilde{w}_n,\ |\lambda_n|^{\frac{5}{2}}w_n \longrightarrow 0 \quad \text{in } L^2(0,1) \tag{127}$$

(for (110), we replace $\frac{i}{\lambda_n^3}w_{nx}$ by $\frac{i}{\lambda_n^6}w_{nx}$ and use (55), and for (111), we use $\frac{1}{|\lambda_n|^{\frac{17}{2}}}$ instead of $\frac{1}{\lambda_n^4}$). Now, we prove some other limits to get (50).

Estimate on w_{nxx} Dividing $(126)_6$ by λ_n^{10} and using (46), (47) and (127), we deduce that

$$(w_{nxx})_n \quad \text{is uniformly bounded in } L^2(0,1). \tag{128}$$

Estimates on φ_{nx}, φ_n and $\widetilde{\varphi}_n$ Taking the inner product of $(126)_6$ with $\frac{\varphi_{nx}}{\lambda_n^9}$ in $L^2(0,1)$, integrating by parts and using (46), (47) and the boundary conditions, we get

$$\begin{aligned} &-\rho_1\left\langle\widetilde{w}_n, \lambda_n\left(i\lambda_n\varphi_{nx} - \widetilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)} + \rho_1\left\langle\lambda_n\widetilde{w}_{nx}, \widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \\ &+k_0\left\langle\lambda_n^2 w_{nx}, \frac{\varphi_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)} + l\left(k + k_0\right)\lambda_n\left\|\varphi_{nx}\right\|_{L^2(0,1)}^2 \\ &+lk\left\langle\lambda_n\left(\psi_n + lw_n\right), \varphi_{nx}\right\rangle_{L^2(0,1)} + \delta\left\langle\lambda_n\theta_{nx}, \varphi_{nx}\right\rangle_{L^2(0,1)} \longrightarrow 0, \end{aligned}$$

hence, using $(126)_1$, (112), (116), (118) and (127), we obtain

$$|\lambda_n|^{\frac{1}{2}}\varphi_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{129}$$

Therefore, according to Poincaré's inequality, (129) leads to

$$|\lambda_n|^{\frac{1}{2}}\varphi_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{130}$$

On the other hand, taking the inner product of $(126)_2$ with $\frac{\varphi_n}{\lambda_n^9}$ in $L^2(0,1)$, integrating by parts and using (46), (47) and the boundary conditions, we get

$$\begin{aligned} &-\rho_1\lambda_n\left\langle\widetilde{\varphi}_n, \left(i\lambda_n\varphi_n - \widetilde{\varphi}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\lambda_n\left\|\widetilde{\varphi}_n\right\|_{L^2(0,1)}^2 \\ &+k\lambda_n\left\langle\left(\varphi_{nx} + \psi_n + lw_n\right), \varphi_{nx}\right\rangle_{L^2(0,1)} - lk_0\lambda_n\left\langle\left(w_{nx} - l\varphi_n\right), \varphi_n\right\rangle_{L^2(0,1)} \longrightarrow 0, \end{aligned}$$

this implies

$$-\rho_1 \left\langle \widetilde{\varphi}_n, \lambda_n \left(i\lambda_n \varphi_n - \widetilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} - \rho_1 \lambda_n \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} + k\lambda_n \left\| \varphi_{nx} \right\|^2_{L^2(0,1)}$$
$$+k \left\langle \left(\lambda_n \psi_n + l\lambda_n w_n \right), \varphi_{nx} \right\rangle_{L^2(0,1)} - lk_0 \left\langle \left(\lambda_n w_{nx} - l\lambda_n \varphi_n \right), \varphi_n \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

so, using $(126)_1$, (112), (127) and (129), we deduce that

$$|\lambda_n|^{\frac{1}{2}} \widetilde{\varphi}_n \longrightarrow 0 \quad \text{in } L^2(0,1), \tag{131}$$

and from $(126)_1$, we obtain that

$$|\lambda_n|^{\frac{3}{2}} \varphi_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{132}$$

Estimates on $\lambda_n \varphi_{nx}$ and $\lambda_n \widetilde{\varphi}_n$ Multiplying $(126)_2$ by $\dfrac{1}{|\lambda_n|^{10+\frac{1}{2}}}$ and using (47), we get

$$i\rho_1 \frac{\lambda_n}{|\lambda_n|^{\frac{1}{2}}} \widetilde{\varphi}_n - k\frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} - k\frac{\psi_{nx}}{|\lambda_n|^{\frac{1}{2}}} - l(k+k_0)\frac{w_{nx}}{|\lambda_n|^{\frac{1}{2}}} + l^2 k_0 \frac{\varphi_n}{|\lambda_n|^{\frac{1}{2}}} \longrightarrow 0 \quad \text{in } L^2(0,1),$$

then, using (46) and (131), we deduce that

$$\frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{133}$$

On the other hand, by integrating by parts and using the boundary conditions, we see that

$$\begin{aligned}
\lambda_n \left\langle w_{nxx}, i\lambda_n \varphi_{nx} \right\rangle_{L^2(0,1)} &= \lambda_n^2 \left\langle i w_{nx}, \varphi_{nxx} \right\rangle_{L^2(0,1)} \\
&= \left\langle \lambda_n \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right), \varphi_{nxx} \right\rangle_{L^2(0,1)} + \lambda_n \left\langle \widetilde{w}_{nx}, \varphi_{nxx} \right\rangle_{L^2(0,1)} \\
&= \left\langle \lambda_n^2 \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right), \frac{\varphi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} + \left\langle \lambda_n |\lambda_n|^{\frac{1}{2}} \widetilde{w}_{nx}, \frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} \right\rangle_{L^2(0,1)},
\end{aligned}$$

then, using (47), $(126)_5$, (127) and (133), we obtain

$$\lambda_n \left\langle w_{nxx}, i\lambda_n \varphi_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{134}$$

Furthermore, integrating by parts and using the boundary conditions, we have

$$\begin{aligned}
\lambda_n \left\langle \left(\varphi_{nx} + \psi_n + l w_n \right)_x, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} &= -\lambda_n \left\langle \left(\varphi_{nx} + \psi_n + l w_n \right), \widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} \\
&= -\frac{1}{lk} \Big\langle \lambda_n^2 \Big[i\lambda_n \rho_1 \widetilde{w}_n - k_0 \left(w_{nx} - l\varphi_n \right)_x \\
&\quad + lk \left(\varphi_{nx} + \psi_n + l w_n \right) + \delta\theta_{nx} \Big], \frac{\widetilde{\varphi}_{nx}}{\lambda_n} \Big\rangle_{L^2(0,1)} \\
&\quad - \frac{1}{lk} \left\langle \left(i\lambda_n \rho_1 \widetilde{w}_n + \delta\theta_{nx} \right), \lambda_n \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx} \right) \right\rangle_{L^2(0,1)} \\
&\quad + \frac{k_0}{lk} \left\langle \left(w_{nx} - l\varphi_n \right)_x, \lambda_n \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx} \right) \right\rangle_{L^2(0,1)} - \frac{\lambda_n^3}{lk} \left\langle i\rho_1 \widetilde{w}_{nx}, i\varphi_n \right\rangle_{L^2(0,1)} \\
&\quad + \frac{\delta}{lk} \left\langle \lambda_n^2 \theta_{nx}, i\varphi_{nx} \right\rangle_{L^2(0,1)} - \frac{k_0 \lambda_n}{lk} \left\langle w_{nxx}, i\lambda_n \varphi_{nx} \right\rangle_{L^2(0,1)} - \frac{k_0 \lambda_n^2}{k} i \left\| \varphi_{nx} \right\|^2_{L^2(0,1)},
\end{aligned}$$

then, using $(126)_1$, $(126)_6$, (127), (128), (132) and (134), we find

$$\lambda_n \left\langle \left(\varphi_{nx} + \psi_n + l w_n \right)_x, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} + \frac{k_0}{k} i \left\| \lambda_n \varphi_{nx} \right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{135}$$

Taking the inner product of (126)$_2$ with $\frac{\widetilde{\varphi}_n}{\lambda_n^9}$ in $L^2(0,1)$ and using (46) and (47s), we get

$$\rho_1 i \left\| \lambda_n \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} - k\lambda_n \left\langle \left(\varphi_{nx} + \psi_n + l w_n\right)_x, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} - l k_0 \left\langle \left(\lambda_n w_{nx} - l\lambda_n \varphi_n\right), \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

then, using (135), we obtain

$$\rho_1 i \left\| \lambda_n \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} + i k_0 \left\| \lambda_n \varphi_{nx} \right\|^2_{L^2(0,1)} - l k_0 \left\langle \left(\lambda_n w_{nx} - l\lambda_n \varphi_n\right), \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

and from (127), (131) and (132), we deduce that

$$\lambda_n \widetilde{\varphi}_n \longrightarrow 0 \quad \text{in } L^2(0,1) \tag{136}$$

and

$$\lambda_n \varphi_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{137}$$

Estimates on ψ_{nx} and $\widetilde{\psi}_n$ and conclusion Taking the inner product of (126)$_2$ with $\frac{\psi_{nx}}{\lambda_n^{10}}$ in $L^2(0,1)$ and using (46) and (47), we get

$$\rho_1 \left\langle i\lambda_n \widetilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} - k \left\langle \varphi_{nxx}, \psi_{nx} \right\rangle_{L^2(0,1)} - k \left\| \psi_{nx} \right\|^2_{L^2(0,1)}$$

$$-l(k+k_0) \left\langle w_{nx}, \psi_{nx} \right\rangle_{L^2(0,1)} + l^2 k_0 \left\langle \varphi_n, \psi_{nx} \right\rangle_{L^2(0,1)} \to 0,$$

then, integrating by parts and using the boundary conditions, we obtain

$$\rho_1 \left\langle i\lambda_n \widetilde{\varphi}_n, \psi_{nx} \right\rangle_{L^2(0,1)} + k \left\langle \lambda_n \varphi_{nx}, \frac{\psi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} - k \left\| \psi_{nx} \right\|^2_{L^2(0,1)}$$
$$-l(k+k_0) \left\langle w_{nx}, \psi_{nx} \right\rangle_{L^2(0,1)} + l^2 k_0 \left\langle \varphi_n, \psi_{nx} \right\rangle_{L^2(0,1)} \to 0,$$

so, using (54), (116), (127), (136) and (137), we deduce that

$$\psi_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{138}$$

Taking the inner product of (126)$_4$ with $\frac{\psi_n}{\lambda_n^{10}}$ in $L^2(0,1)$, integrating by parts and using (46), (47) and the boundary conditions, we get

$$-\rho_2 \left\langle \widetilde{\psi}_n, \left(i\lambda_n \psi_n - \widetilde{\psi}_n\right) \right\rangle_{L^2(0,1)} - \rho_2 \left\| \widetilde{\psi}_n \right\|^2_{L^2(0,1)} + b \left\| \psi_{nx} \right\|^2_{L^2(0,1)}$$
$$+ \left\langle k\left(\varphi_{nx} + \psi_n + l w_n\right), \psi_n \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

hence, using (54), (126)$_3$ and (138), we get

$$\widetilde{\psi}_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{139}$$

A combination of the limits (54), (118), (127), (136), (138) and (139) leads to (50), which is a contradiction with (46). Consequently, (103) with $j = 10$ holds.

4.3 Case of system (4) with $\rho_1 b - \rho_2 k = 0$

The limit (105) with $j = 4$ implies that

$$\begin{cases} \lambda_n^4 \left[i\lambda_n \varphi_n - \widetilde{\varphi}_n \right] \to 0 \quad \text{in } H_*^1(0,1), \\ \lambda_n^4 \left[i\rho_1 \lambda_n \widetilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + l w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) \right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n \psi_n - \widetilde{\psi}_n \right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_2 \lambda_n \widetilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l w_n\right) \right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n w_n - \widetilde{w}_n \right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_1 \lambda_n \widetilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l w_n\right) + \delta \widetilde{\theta}_{nx} \right] \to 0 \quad \text{in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n \theta_n - \widetilde{\theta}_n \right] \to 0 \quad \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_3 \lambda_n \theta_n - \beta\left(\theta_n + \gamma \widetilde{\theta}_n\right)_{xx} + \delta \widetilde{w}_{nx} \right] \to 0 \quad \text{in } L^2(0,1). \end{cases} \tag{140}$$

Estimates on $\theta_{nx}, \theta_n, \widetilde{\theta}_{nx}$ and $\widetilde{\theta}_n$ and conclusion Taking the inner product of $\lambda_n^4 \left(i\,\lambda_n\, I - \mathcal{A}\right) \Phi_n$ with Φ_n in $\mathcal{H}$ and using (17), we get

$$\begin{aligned} Re\left\langle \lambda_n^4 \left(i\,\lambda_n\, I - \mathcal{A}\right) \Phi_n, \Phi_n \right\rangle_{\mathcal{H}} &= Re\left(i\lambda_n^5 \left\|\Phi_n\right\|_{L^2(0,1)}^2 + \gamma \lambda_n^4 \left\|\widetilde{\theta}_{nx}\right\|_{L^2(0,1)}^2 \right) \\ &= \gamma \lambda_n^4 \left\|\widetilde{\theta}_{nx}\right\|_{L^2(0,1)}^2 . \end{aligned}$$

So (46) and (105) imply that

$$\lambda_n^2 \widetilde{\theta}_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{141}$$

Because θ_n in $H_*^1(0,1)$ and thanks to Poincaré's inequality, we deduce that

$$\lambda_n^2 \widetilde{\theta}_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{142}$$

Multiplying $(140)_7$ by $\dfrac{1}{\lambda_n^2}$ and using (46), (47), (141) and (142), we have

$$\lambda_n^3 \theta_{nx} \longrightarrow 0 \quad \text{in } L^2(0,1) \tag{143}$$

and

$$\lambda_n^3 \theta_n \longrightarrow 0 \quad \text{in } L^2(0,1), \tag{144}$$

so (107) and (108) hold. Consequently, the proof can be ended exactly as in case of system (1) with $j = 4$ (Sect. 4.1).

4.4 Case of system (4) with $\rho_1 b - \rho_2 k \neq 0$

The limit (105) with $j = 10$ implies (140) with λ_n^{10} instead of λ_n^4. Similar calculations as in the case of system (1) with $\rho_1 b - \rho_2 k \neq 0$ (Sect. 4.2) give the desired result. We omit the details.

Hence, the proof of our Theorem 4.1 is completed. □

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