STABILITY OF A TRANSLATIONAL TIMOSHENKO SYSTEM IN THERMOELASTICITY WITH SECOND SOUND

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Abstract. We consider a Timoshenko system coupled with heat equations modelled by Cattaneo's law. The coupling is through the transverse displacement. Both ends of the beam are dynamic. One end of the beam is fixed to a base in a translational motion and a tip mass is attached to the other end. We design a feedback control acting at the base. It is shown that this feedback control is a reasonable one and is capable of stabilizing the system. We prove an exponential and a polynomial stability result using the multiplier technique. To this end, we introduce new functionals to form a suitable Lyapunov functional.

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1. Introduction

A translational beam modelled as a Timoshenko system and coupled through its transversal displacement component to two equations resulting from Cattaneo's law, is considered here. One of its end is attached to a base which moves in a translational manner and a dynamic mass is attached to the free end of the beam. The dynamic of the structure is modelled by five differential equations: two partial differential equations accounting for the Timoshenko system, two partial differential equations obeying Cattaneo's law and one ordinary differential equation modelling the dynamic of the base. More precisely, we consider the model

$$(1.1)\qquad \begin{cases} m(S_{tt}(t)+\varphi_{tt}(0,t))+\int_0^L \rho_1\left(S_{tt}(t)+\varphi_{tt}(x,t)\right)dx+m_E\left(S_{tt}(t)+\varphi_{tt}(L,t)\right)=\tau\left(t\right), \\ \rho_1\left(S_{tt}(t)+\varphi_{tt}(x,t)\right)-k\left(\varphi_x(x,t)+\psi(x,t)\right)_x+\gamma\theta_x(x,t)=0, \\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+k\left(\varphi_x(x,t)+\psi(x,t)\right)=0, \\ \rho_3\theta_t(x,t)+\bar{\sigma}_x(x,t)+\gamma\varphi_{tx}(x,t)=0, \\ \tau_0\bar{\sigma}_t(x,t)+\delta\bar{\sigma}(x,t)+\kappa\theta_x(x,t)=0 \end{cases}$$

with the boundary conditions

$$(1.2)\qquad \begin{cases} \varphi_x(0,t)=\psi(0,t)=\theta(0,t)=\theta(L,t)=0, \\ k(\varphi_x(L,t)+\psi(L,t))+m_E\left(S_{tt}(t)+\varphi_{tt}(L,t)\right)+\mu\left(S_t(t)+\varphi_t(L,t)\right)=0, \\ b\psi_x(L,t)+J\psi_{tt}(L,t)=0 \end{cases}$$

and the initial data

$$(1.3)\qquad \begin{cases} \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\ S(0)=S_0,\ S_t(0)=S_1, \\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \bar{\sigma}(x,0)=\bar{\sigma}_0(x),\ \theta(x,0)=\theta_0(x), \end{cases}$$

where $x\in[0,L]$ and $t\in\mathbb{R}^+$. Here, φ is the beam transversal displacement, ψ is the rotation angle of the beam, θ is the temperature difference, $\bar{\sigma}$ is the heat flux vector obeying Cattaneo's law, S is the base

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motion displacement while τ is the feedback control. All the parameters are positive constants; ρ_1 is the mass density, ρ_2 is the moment mass inertia, m is the mass of the translational base, m_E is the mass with rotational J attached at the free end of the beam, b is the rigidity coefficient (of the cross-section), k is the shear modulus of elasticity and L is the length of the beam. The constants $\rho_3, \gamma, \tau_0, \delta$ and κ relate to hypotheses in thermoelasticity.

This model can derived easily from the simple Timoshenko beam model in translational movement by taking into account the second sound. That is we modify the Hamilton principle

$$\delta \int_{t_a}^{t_b} [E_k(t) - E_p(t) + W(t)]\, dt = 0,$$

where $E_k(t)$ is the kinetic energy

$$E_k(t) := \frac{m}{2} S_t^2(t) + \frac{\rho_1}{2} \|S_t + \varphi_t\|^2 + \frac{\rho_2}{2} \|\psi_t\|^2 + m_E\, [S_t(t) + \varphi_t(L,t)]^2 + \frac{J}{2} \psi_t^2(L,t),$$

($\|\cdot\|$ here is the L^2-norm) and $E_p(t)$ is the potential energy

$$E_p(t) := \frac{b}{2} \|\psi_x\|^2 + k \|\varphi_x + \psi\|^2 ,$$

whilst the virtual work done by $\tilde{\tau}(t)$ (external force) is

$$\delta W(t) := \tilde{\tau}(t) \delta S(t).$$

In view of the second sound, we add the expressions $\|\theta\|^2$ and $\|\bar{\sigma}\|^2$. Then, applying the variational operator, integrating (by parts) and using the appropriate boundary conditions, we end up with (1.1).

The system, as it is, is unstable. It is the purpose of this work to find a control to be applied to the base so as to stabilize the system preferably in an exponential fashion.

We are not aware of similar works for beams in translational movement. Without translation, however, that is with $S \equiv 0$, we can find a fairly big number of papers, let alone Timoshenko systems with other types of damping (frictional, structural, viscoelastic) different from the thermal one [1], [7], [8], [10], [13], [14], [15], [16], [20] and [21]. Many valuable papers may be found in the references of the papers cited here.

For this particular type of thermoelasticity with second sound, we may distinguish two types of coupling: the coupling through the rotation angle of the beam and the coupling through the beam transversal displacement. The first case has been studied in [9] where it has been shown that the system is not exponentially stable when $\tau_0 \neq 0$ (that is when we are not in the classical thermoelastic case). In [18], the authors came up with a 'stability number' and proved exponential stability in case this number is zero. In case this number is not zero, there is no exponential stability but rather polynomial stability. To stabilize the system, some authors have added extra damping (frictional or viscoelastic) like in the paper [13].

We are interested in the second case $S \neq 0$. A closely related work to ours is in [2] where the authors discussed the stability of the system without extra damping. However, they considered homogeneous Dirichlet boundary conditions for the rotation angle of the beam and temperature difference whereas the boundary conditions of the transversal displacement is of Neumann type, but again with $S \equiv 0$ and without end mass.

In the present paper the situation is even more delicate. Indeed, we have a beam in translational movement. In fact, both endpoints are dynamic. Mathematically, the type of boundary conditions at L in (1.2) are not easy to deal with. In addition to, searching for an appropriate reasonable control to be implemented at the base, one needs to handle the arising boundary terms. To this end, we have been forced to come up with certain suitable functionals. These functionals are added to the classical energy functional, in addition to some standard ones, with the aim to obtain an appropriate equivalent one to work with. As a matter of fact, a common practice is to make a transformation so as to have zero means. This allows us to use the Poincaré's inequality (in case it cannot be applied to the original state) and also to get rid of some boundary terms. In fact, our functionals are chosen, partly, to lead to this situation.

The interest to beams subject to translational movement was motivated by works of engineers; see for instance [6], [11], [22] and others. In particular, based on the analysis in [6], the present second and third authors have initiated a series of papers on this important subject [3], [4], [5] and [12].

In the next section we transform the problem into a simpler one, determine a feedback control τ (see (2.10) below), the energy, compute its derivative and show that the system is dissipative and well posed. Several functionals are introduced in Section 3. Moreover, we prove an exponential stability in case a 'stability number' (defined in (3.11)) is equal to zero. A polynomial stability result is proved in Section 4 in case this number is not zero. We end our paper by giving some general remarks in the last section.

2. Control, energy and well-posedness

In this section we first transform our problem into a simpler one. Then, we define the energy of the system and suggest a reasonable feedback control. It is then proved that the energy is decaying (but without an explicit rate at this stage).

We start by defining the total deflection of the beam as follows:

$$\chi(x,t) = S(t) + \varphi(x,t). \tag{2.1}$$

By combining the second equation in (1.1) and the boundary conditions $(1.2)_1$, we get

$$\int_0^L \rho_1 \left(S_{tt}(t) + \varphi_{tt}(x,t)\right) dx - k\left(\varphi_x(L,t) + \psi(L,t)\right) = 0,$$

so, using the boundary conditions $(1.2)_2$, we find

$$\int_0^L \rho_1 \left(S_{tt}(t) + \varphi_{tt}(x,t)\right) dx + m_E\left(S_{tt}(t) + \varphi_{tt}(L,t)\right) = -\mu\left(S_t(t) + \varphi_t(L,t)\right).$$

Then, with this new function χ defined in (2.1), the problem (1.1)-(1.3) is equivalent to

$$\begin{cases} m\chi_{tt}(0,t) - \mu\chi_t(L,t) = \tau(t), \\ \rho_1\chi_{tt}(x,t) - k\left(\chi_x(x,t) + \psi(x,t)\right)_x + \gamma\theta_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k\left(\chi_x(x,t) + \psi(x,t)\right) = 0, \\ \rho_3\theta_t(x,t) + \bar{\sigma}_x(x,t) + \gamma\chi_{tx}(x,t) = 0, \\ \tau_0\bar{\sigma}_t(x,t) + \delta\bar{\sigma}(x,t) + \kappa\theta_x(x,t) = 0 \end{cases} \tag{2.2}$$

with the boundary conditions

$$\begin{cases} \chi_x(0,t) = \psi(0,t) = \theta(0,t) = \theta(L,t) = 0, \\ k(\chi_x(L,t) + \psi(L,t)) + m_E\chi_{tt}(L,t) + \mu\chi_t(L,t) = 0, \\ b\psi_x(L,t) + J\psi_{tt}(L,t) = 0 \end{cases} \tag{2.3}$$

and the initial data

$$\begin{cases} \chi(x,0) = S_0 + \varphi_0(x) =: \chi_0(x),\ \chi_t(x,0) = S_1 + \varphi_1(x) =: \chi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x),\ \bar{\sigma}(x,0) = \bar{\sigma}_0(x),\ \theta(x,0) = \theta_0(x). \end{cases} \tag{2.4}$$

Before we move to the energy, its derivative and the control, we make another transformation. This will allow the use of Poincaré's inequality and also allows us to deal with certain boundary terms. Observe that, from the last equation in (2.2) and the boundary conditions (2.3), we have

$$\frac{d}{dt}\int_0^L \bar{\sigma}(x,t)dx = -\frac{\delta}{\tau_0}\int_0^L \bar{\sigma}(x,t)dx,$$

and therefore, by integrating and using the initial data (2.4),

$$\int_0^L \bar{\sigma}(x,t)dx = e^{-\frac{\delta t}{\tau_0}}\int_0^L \bar{\sigma}_0(x)dx. \tag{2.5}$$

Putting

$$\sigma(x,t) = \bar{\sigma}(x,t) - \frac{1}{L}e^{-\frac{\delta t}{\tau_0}}\int_0^L \bar{\sigma}_0(x)dx, \tag{2.6}$$

we see that (2.5) and (2.6) lead to

$$\int_0^L \sigma(x,t)dx = 0. \tag{2.7}$$

Profiting from the above property (2.7), the Poincaré's inequality is applicable to σ. In addition, it may be easily verified that $(\chi, \psi, \theta, \sigma)$ satisfies the system

$$\begin{cases} m\chi_{tt}(0,t) - \mu\chi_t(L,t) = \tau(t), \\ \rho_1\chi_{tt}(x,t) - k\left(\chi_x(x,t) + \psi(x,t)\right)_x + \gamma\theta_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k\left(\chi_x(x,t) + \psi(x,t)\right) = 0, \\ \rho_3\theta_t(x,t) + \sigma_x(x,t) + \gamma\chi_{tx}(x,t) = 0, \\ \tau_0\sigma_t(x,t) + \delta\sigma(x,t) + \kappa\theta_x(x,t) = 0 \end{cases} \tag{2.8}$$

with the boundary conditions (2.3) and the initial data

$$\begin{cases} \chi(x,0) = \chi_0(x),\ \chi_t(x,0) = \chi_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ \sigma(x,0) = \bar{\sigma}_0(x) - \frac{1}{L}\int_0^L \bar{\sigma}_0(x)dx =: \sigma_0(x),\ \theta(x,0) = \theta_0(x). \end{cases} \tag{2.9}$$

We suggest the feedback control force

$$\tau(t) = -K\chi_t(0,t) - \mu\chi_t(L,t) - \chi(0,t), \tag{2.10}$$

where K is a positive 'control gain'. This kind of feedback control is dictated by the calculations in this method. In particular, it is needed for the dissipativity of the energy and in the derivation of the differential inequalities satisfied by $V_1(t)$ and $V_2(t)$ (defined in Sections 3 and 4, respectively).

The problem (2.8) with the initial data (2.9) and the control (2.10) can be written in the form

$$\begin{cases} \Psi_t = \mathcal{B}\Psi, \\ \Psi(t=0) = \Psi_0, \end{cases} \tag{2.11}$$

for $\Psi := (\chi, w, \psi, z, \theta, \sigma, \xi, \eta, y)^T$ and $\Psi_0 := (\chi_0, \chi_1, \psi_0, \psi_1, \theta_0, \sigma_0, \chi_1(0,\cdot), \psi_1(L,\cdot), \chi_1(L,\cdot))^T$ with $w = \chi_t$, $z = \psi_t$, $\xi = \chi_t(0,\cdot)$, $\eta = \psi_t(L,\cdot)$, $y = \chi_t(L,\cdot)$ and

$$\mathcal{B}\Psi = \begin{pmatrix} w \\ \frac{1}{\rho_1}\left[k\left(\chi_x + \psi\right)_x - \gamma\theta_x\right] \\ z \\ \frac{1}{\rho_2}\left[b\psi_{xx} - k\left(\chi_x + \psi\right)\right] \\ -\frac{1}{\rho_3}\left(\sigma_x + \gamma w_x\right) \\ -\frac{1}{\tau_0}\left(\delta\sigma + \kappa\theta_x\right) \\ -\frac{1}{m}\left(K\xi + \chi(0,\cdot)\right) \\ -\frac{b}{J}\psi_x(L,\cdot) \\ -\frac{1}{m_E}\left[k(\chi_x(L,\cdot) + \psi(L,\cdot)) + \mu y\right] \end{pmatrix}. \tag{2.12}$$

Taking into consideration (2.7) and the Dirichlet boundary conditions in (2.3), we introduce the spaces

$$L_*^2(0,L) := \left\{ f \in L^2(0,L) : \int_0^L f(x)dx = 0 \right\},$$

$$H_*^1(0,L) := H^1(0,L) \cap L_*^2(0,L), \quad V_0(0,L) := \left\{ f \in H^1(0,L) : f(0) = 0 \right\}$$

and

$$\mathcal{H} := H^1(0,L) \times L^2(0,L) \times V_0(0,L) \times L^2(0,L) \times L^2(0,L) \times L_*^2(0,L) \times \mathbb{R}^3.$$

The Neumann boundary conditions in (2.3) are considered in the definition of the domain of $\mathcal{B}$ given by

$$\begin{aligned} D(\mathcal{B}) := \quad & \Big\{ \Psi \in \mathcal{H} : \chi \in H_*^2(0,L), \psi \in H^2(0,L) \cap V_0(0,L), w \in H^1(0,L), z \in V_0(0,L), \\ & \theta \in H_0^1(0,L), \sigma \in H_*^1(0,L), \xi = w(0,\cdot), \eta = z(L), y = w(L) \Big\}, \end{aligned}$$

where

$$H_*^2(0,L) := \left\{ f \in H^2(0,L) : f_x(0) = 0 \right\}.$$

Then (2.11) is a convenient formulation of (2.3), (2.8) and (2.9). The space $\mathcal{H}$ is a Hilbert space, where for

$$\Psi = (\chi, w, \psi, z, \theta, \sigma, \xi, \eta, y)^T \quad \text{and} \quad \tilde{\Psi} = \left(\tilde{\chi}, \tilde{w}, \tilde{\psi}, \tilde{z}, \tilde{\theta}, \tilde{\sigma}, \tilde{\xi}, \tilde{\eta}, \tilde{y}\right)^T,$$

the inner product of $\mathcal{H}$ is given by

$$\begin{aligned} \langle \Psi, \tilde{\Psi} \rangle_{\mathcal{H}} &= k\langle \chi_x + \psi, \tilde{\chi}_x + \tilde{\psi} \rangle + b\langle \psi_x, \tilde{\psi}_x \rangle + \rho_3 \langle \theta, \tilde{\theta} \rangle + \frac{\tau_0}{\kappa} \langle \sigma, \tilde{\sigma} \rangle \\ &\quad + \rho_1 \langle w, \tilde{w} \rangle + \rho_2 \langle z, \tilde{z} \rangle + m\xi\tilde{\xi} + J\eta\tilde{\eta} + m_E y\tilde{y} + \chi(0)\tilde{\chi}(0). \end{aligned} \tag{2.13}$$

Here, $\langle \cdot, \cdot \rangle$ is the standard inner product of $L^2(0, L)$. The associated energy to (2.3), (2.8) and (2.9) is given by

$$E_1(t) := \frac{1}{2} \|\Psi\|^2_{\mathcal{H}}; \tag{2.14}$$

that is (we denote by $\|\cdot\|_2$ the classical norm of $L^2(0, L)$)

$$\begin{aligned} E_1(t) &= \frac{1}{2}\left[k\|\chi_x + \psi\|_2^2 + b\|\psi_x\|_2^2 + \rho_3\|\theta\|_2^2 + \frac{\tau_0}{\kappa}\|\sigma\|_2^2 + \rho_1\|\chi_t\|_2^2 + \rho_2\|\psi_t\|_2^2\right] \\ &\quad + \frac{1}{2}\left[m\chi_t^2(0, t) + J\psi_t^2(L, t) + m_E\chi_t^2(L, t) + \chi^2(0, t)\right]. \end{aligned} \tag{2.15}$$

We see that, if $\|\Psi\|_{\mathcal{H}} = 0$, then

$$\chi(0) = \chi_t(L) = \psi_t(L) = \chi_t(0) = \psi_t = \chi_t = \sigma = \theta = \psi_x = \chi_x + \psi = 0, \tag{2.16}$$

which implies that $\chi_x = -\psi$ and ψ is a constant function with respect to x. According to the homogeneous Dirichlet boundary conditions in (2.3), we deduce that $\psi = 0$ and χ is a constant function with respect to x. But from (2.16) we have $\chi(0) = 0$, so $\chi = 0$. Consequently, $\Psi = 0$. This implies that (2.13) generates a norm on $\mathcal{H}$, and then $\mathcal{H}$ endowed with (2.13) is a Hilbert space.

Now, multiplying the second equation in (2.8) by $\chi_t(x, t)$ and integrating over $[0, L]$, we find after using the boundary conditions (2.3)

$$\frac{d}{dt}\left[\frac{\rho_1}{2}\|\chi_t\|_2^2 + \frac{m_E}{2}\chi_t^2(L, t)\right] + k\langle \chi_{xt}, \chi_x + \psi \rangle + \gamma\langle \theta_x, \chi_t \rangle + \mu\chi_t^2(L, t) = 0. \tag{2.17}$$

Next, we multiply the third equation in (2.8) by $\psi_t(x, t)$ and integrate over $[0, L]$, we get

$$\frac{d}{dt}\left[\frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2\right] - b\psi_t(L, t)\,\psi_x(L, t) + k\langle \psi_t, \chi_x + \psi \rangle = 0$$

or, using the boundary conditions (2.3),

$$\frac{d}{dt}\left[\frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2\right] + J\psi_{tt}(L, t)\psi_t(L, t) + k\langle \psi_t, \chi_x + \psi \rangle = 0.$$

This identity may be rewritten as

$$\frac{d}{dt}\left[\frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{b}{2}\|\psi_x\|_2^2 + \frac{J}{2}\psi_t^2(L, t)\right] + k\langle \psi_t, \chi_x + \psi \rangle = 0. \tag{2.18}$$

Continuing in the same way (multiply the fourth equation in (2.8) by $\theta(x, t)$), we obtain

$$\frac{d}{dt}\left[\frac{\rho_3}{2}\|\theta\|_2^2\right] + \langle \sigma_x, \theta \rangle + \gamma\langle \chi_{tx}, \theta \rangle = 0,$$

so, by integrating by parts and using the boundary conditions (2.3), we find

$$\frac{d}{dt}\left[\frac{\rho_3}{2}\|\theta\|_2^2\right] = \langle \sigma, \theta_x \rangle + \gamma\langle \chi_t, \theta_x \rangle. \tag{2.19}$$

The last equation in (2.8) (multiplied by $\sigma(x, t)$) gives

$$\frac{d}{dt}\left[\frac{\tau_0}{2\kappa}\|\sigma\|_2^2\right] = -\frac{\delta}{\kappa}\|\sigma\|_2^2 - \langle \sigma, \theta_x \rangle. \tag{2.20}$$

Finally, multiplying the first equation in (2.8) by $\chi_t(0, t)$ and using (2.10) leads to

$$\frac{d}{dt}\left[\frac{m}{2}\chi_t^2(0, t) + \frac{1}{2}\chi^2(0, t)\right] = -K\chi_t^2(0, t). \tag{2.21}$$

By virtue of the relations (2.15)-(2.21), we obtain

$$E_1'(t) = -K\chi_t^2(0,t) - \mu\chi_t^2(L,t) - \frac{\delta}{\kappa}\|\sigma\|_2^2 . \tag{2.22}$$

Theorem 1. Assuming the initial data $\Psi_0 \in \mathcal{H}$, there exists a unique mild (weak) solution of (2.11) satisfying

$$\Psi \in C\left(\mathbb{R}^+, \mathcal{H}\right). \tag{2.23}$$

In case $\Psi_0 \in D(\mathcal{B})$, the solution is classical; that is,

$$\Psi \in C\left(\mathbb{R}^+, D(\mathcal{B})\right) \cap C^1\left(\mathbb{R}^+, \mathcal{H}\right). \tag{2.24}$$

Proof. The well-posedness may be derived easily in a standard way (see also [19]). We give here a brief sketch of the proof. First, (2.11), (2.14) and (2.22) lead to

$$\langle \mathcal{B}\Psi, \Psi\rangle_{\mathcal{H}} = -\left[K\chi_t^2(0,t) + \mu\chi_t^2(L,t) + \frac{\delta}{\kappa}\|\sigma\|_2^2\right] \le 0, \tag{2.25}$$

for any $\Psi \in D(\mathcal{B})$, hence $\mathcal{B}$ is a dissipative operator.

Second, we show that $I - \mathcal{B}$ is surjective, where I denotes the identity operator. Let

$$F = (f_1, \cdots, f_9) \in \mathcal{H}.$$

We claim that there exists $\Psi \in D(\mathcal{B})$ satisfying

$$(I - \mathcal{B})\Psi = F. \tag{2.26}$$

Using the definition (2.12) of $\mathcal{B}$, the first, third and last three equations in (2.26) are equivalent to

$$\begin{cases} w = \chi - f_1, \\ z = \psi - f_3, \\ \xi = \frac{1}{K+m}(mf_7 - \chi(0)), \\ \eta = f_8 - \frac{b}{J}\psi_x(L), \\ y = \frac{m_E}{m_E+\mu} f_9 - \frac{k}{m_E+\mu}(\chi_x(L) + \psi(L)). \end{cases} \tag{2.27}$$

Consequently, if $\chi \in H_*^2(0,L)$ and $\psi \in H^2(0,L) \cap V_0(0,L)$, then w, z, ξ, η and y exist and

$$(w, z) \in H^1(0,L) \times V_0(0,L).$$

Moreover, $\xi = w(0)$, $\eta = z(L)$ and $y = w(L)$ if

$$\begin{cases} \chi(0) = \frac{1}{K+m+1}\left[(K+m)f_1(0) + mf_7\right] := g_1, \\ \psi(L) + \frac{b}{J}\psi_x(L) = f_3(L) + f_8 := g_2, \\ \frac{k}{m_E+\mu}(\chi_x(L) + \psi(L)) + \chi(L) = f_1(L) + \frac{m_E}{m_E+\mu} f_9 := g_3. \end{cases} \tag{2.28}$$

On the other hand, if $\sigma \in H_*^1(0,L)$, then the function

$$\theta(x) = \frac{1}{k}\int_0^x \left[\tau_0 f_6(s) - (\tau_0 + \delta)\sigma(s)\right] ds \tag{2.29}$$

satisfies the sixth equation in (2.26) with $\theta \in H_0^1(0,L)$. We put

$$\hat{\chi} = \chi - g_1 \quad \text{and} \quad \hat{\sigma}(x) = \int_0^x \sigma(s)ds.$$

Observe that, if $\hat{\chi} \in H_*^2(0,L) \cap V_0(0,L)$, then $\chi \in H_*^2(0,L)$ and $\chi(0) = g_1$ (notice that g_1 is a constant). Moreover, using (2.29) and the first relation in (2.27), we remark that the fifth equation in (2.26) holds with $\sigma = \hat{\sigma}_x \in H_*^1(0,L)$ if $\chi \in H_*^2(0,L)$ and the equation

$$\frac{\rho_3(\tau_0 + \delta)}{k}\hat{\sigma} - \hat{\sigma}_{xx} = \gamma(\chi_x - f_{1x}) - \rho_3 f_5 + \frac{\rho_3\tau_0}{k}\int_0^x f_6(s)ds \tag{2.30}$$

has a solution $\hat{\sigma} \in H^2(0,L) \cap H_0^1(0,L)$. According to (2.27), (2.28), (2.29) and (2.30), we see that the second, fourth and fifth equations in (2.26) can be reduced to

$$\begin{cases} \rho_1 \hat{\chi} - k\left(\hat{\chi}_x + \psi\right)_x - \frac{\gamma(\tau_0+\delta)}{k}\hat{\sigma}_x = \rho_1(f_1+f_2-g_1) - \frac{\gamma\tau_0}{k} f_6 := h_1, \\ \rho_2 \psi - b\psi_{xx} + k\left(\hat{\chi}_x + \psi\right) = \rho_2(f_3+f_4) := h_2, \\ \frac{\rho_3(\tau_0+\delta)^2}{k^2}\hat{\sigma} - \frac{\tau_0+\delta}{k}\hat{\sigma}_{xx} - \frac{\gamma(\tau_0+\delta)}{k}\hat{\chi}_x = \frac{\tau_0+\delta}{k}\left[-\gamma f_{1x} - \rho_3 f_5 + \frac{\rho_3\tau_0}{k}\int_0^x f_6(s)ds\right] := h_3. \end{cases} \tag{2.31}$$

Then, (2.26) has a solution $\Psi \in D(\mathcal{B})$ if (2.31) has a solution

$$(\hat{\chi}, \psi, \hat{\sigma}) \in (H^2(0,L) \cap V_0(0,L)) \times (H^2(0,L) \cap V_0(0,L)) \times (H^2(0,L) \cap H_0^1(0,L)) \tag{2.32}$$

satisfying

$$\begin{cases} \hat{\chi}_x(0) = 0, \\ \psi(L) + \frac{b}{J}\psi_x(L) = g_2, \\ \frac{k}{m_E+\mu}(\hat{\chi}_x(L) + \psi(L)) + \hat{\chi}(L) = g_3 - g_1. \end{cases} \tag{2.33}$$

To this end, we consider the variational formulation of (2.31) in

$$\bar{H} := V_0(0,L) \times V_0(0,L) \times H_0^1(0,L).$$

By multiplying the equations in (2.31) by the test functions $\tilde{\chi} \in V_0(0,L)$, $\tilde{\psi} \in V_0(0,L)$ and $\tilde{\sigma} \in H_0^1(0,L)$, respectively, integrating by parts, using (2.33) and adding the obtained formulas, we get

$$a((\hat{\chi}, \psi, \hat{\sigma}), (\tilde{\chi}, \tilde{\psi}, \tilde{\sigma})) = l(\tilde{\chi}, \tilde{\psi}, \tilde{\sigma}), \quad \forall (\tilde{\chi}, \tilde{\psi}, \tilde{\sigma}) \in \bar{H}, \tag{2.34}$$

where

$$\begin{aligned} a((\hat{\chi}, \psi, \hat{\sigma}), (\tilde{\chi}, \tilde{\psi}, \tilde{\sigma})) \quad = \quad & k\langle \hat{\chi}_x + \psi, \tilde{\chi}_x + \tilde{\psi}\rangle + b\langle \psi_x, \tilde{\psi}_x\rangle + \frac{\tau_0+\delta}{k}\langle \hat{\sigma}_x, \tilde{\sigma}_x\rangle + \rho_1\langle \hat{\chi}, \tilde{\chi}\rangle \\ & + \rho_2\langle \psi, \tilde{\psi}\rangle + \frac{\rho_3(\tau_0+\delta)^2}{k^2}\langle \hat{\sigma}, \tilde{\sigma}\rangle + \frac{\gamma(\tau_0+\delta)}{k}(\langle \hat{\chi}, \tilde{\sigma}_x\rangle \\ & - \langle \hat{\sigma}_x, \tilde{\chi}\rangle) + (m_E+\mu)\hat{\chi}(L)\tilde{\chi}(L) + J\psi(L)\tilde{\psi}(L) \end{aligned}$$

and

$$l(\tilde{\chi}, \tilde{\psi}, \tilde{\sigma}) = \langle h_1, \tilde{\chi}\rangle + \langle h_2, \tilde{\psi}\rangle + \langle h_3, \tilde{\sigma}\rangle + (m_E+\mu)(g_3-g_1)\tilde{\chi}(L) + Jg_2\tilde{\psi}(L).$$

It is easy to see that a is a bilinear, continuous and coercive form on $\bar{H} \times \bar{H}$. Moreover, because $h_1, h_2, h_3 \in L^2(0,L)$ and g_1, g_2 and g_3 are constants, l is a linear and continuous form on $\bar{H}$. Then, Lax-Milgram theorem implies that (2.34) admits a unique solution $(\hat{\chi}, \psi, \hat{\sigma}) \in \bar{H}$, and hence, by classical regularity arguments, we deduce that $(\hat{\chi}, \psi, \hat{\sigma})$ solves (2.31) and satisfies (2.32) and (2.33). This proves that (2.26) has a solution $\Psi \in D(\mathcal{B})$.

Finally, the linear operator $\mathcal{B}$ is maximal monotone, and then it generates a linear C_0-semigroup of contractions on $\mathcal{H}$ and $D(\mathcal{B})$ is dense in $\mathcal{H}$. So, Theorem 1 holds thanks to Hille-Yosida theorem.

3. Exponential stability

In this section, we start by considering classical solutions, so the introduced functionals are well differentiable and all the computations are justified. Our stability result (3.30) can then be extended to mild solutions by density and continuity arguments, since the constants α and β in (3.30) depend continuously on $\|\Psi_0\|_{\mathcal{H}}$.

Here, we introduce functionals U_j, $j = 1, \cdots, 8$, to be added (with certain weights) to the energy E_1 in order to obtain a new Lyapunov functional (equivalent to the energy; see (3.28) below) leading to an exponential stability.

Let ω be the solution of

$$-\omega_{xx} = \psi_x, \quad \omega_x(0) = \omega(L) = 0.$$

This function ω can be explicitly given in term of ψ by

$$\omega(x,t) = -\int_0^x \psi(\lambda,t)d\lambda + \int_0^L \psi(\lambda,t)d\lambda. \tag{3.1}$$

Lemma 1. The derivative of the following functional:

$$U_1(t) := \int_0^L (\rho_1 \chi_t \omega + \rho_2 \psi_t \psi)\, dx + J\psi_t(L,t)\psi(L,t)$$

along solutions of (2.3), (2.8) and (2.9) satisfies, for any $\varepsilon_1, \varepsilon_3 > 0$,

$$U_1'(t) \leq \left(\frac{\gamma C_p}{4\varepsilon_3} - b\right) \|\psi_x\|_2^2 + \left(\frac{\rho_1 C_p}{4\varepsilon_1} + \rho_2\right) \|\psi_t\|_2^2 + J\psi_t^2(L,t) + \gamma\varepsilon_3 \|\theta\|_2^2 + \rho_1\varepsilon_1 \|\chi_t\|_2^2, \tag{3.2}$$

where C_p is a positive constant depending only on L and Poincaré's constant.

Proof. From the definition (3.1) of ω, we have $-\omega_x = \psi$. Then, differentiating the first term in $U_1(t)$ and using the boundary conditions at $x = 0$ on χ and ψ, and at $x = L$ on ω, we find

$$\begin{aligned}
\rho_1 \frac{d}{dt}\int_0^L \chi_t \omega dx &= \rho_1 \int_0^L (\chi_{tt}\omega + \chi_t\omega_t)\, dx \\
&= \int_0^L [k(\chi_x + \psi)_x - \gamma\theta_x]\, \omega dx + \rho_1 \int_0^L \chi_t \omega_t dx \\
&= -k\int_0^L (\chi_x + \psi)\,\omega_x dx - \gamma \int_0^L \theta_x \omega dx + \rho_1 \int_0^L \chi_t\omega_t dx \\
&= k\int_0^L (\chi_x + \psi)\,\psi dx + \gamma\int_0^L \theta\omega_x dx + \rho_1\int_0^L \chi_t\omega_t dx \\
&= k\int_0^L (\chi_x + \psi)\,\psi dx - \gamma\int_0^L \theta\psi dx + \rho_1\int_0^L \chi_t\omega_t dx.
\end{aligned}$$

The derivative of the second term of $U_1(t)$ is equal to

$$\begin{aligned}
\rho_2 \frac{d}{dt}\int_0^L \psi_t\psi dx &= \rho_2 \int_0^L \left(\psi_{tt}\psi + \psi_t^2\right) dx \\
&= \rho_2 \|\psi_t\|_2^2 + b\int_0^L \psi_{xx}\psi dx - k\int_0^L (\chi_x + \psi)\,\psi dx \\
&= \rho_2 \|\psi_t\|_2^2 + b\,[\psi_x\psi]_0^L - b\|\psi_x\|_2^2 - k\int_0^L (\chi_x + \psi)\,\psi dx \\
&= \rho_2 \|\psi_t\|_2^2 - J\psi_{tt}(L,t)\psi(L,t) - b\|\psi_x\|_2^2 - k\int_0^L (\chi_x + \psi)\,\psi dx.
\end{aligned}$$

Therefore

$$U_1'(t) = -b\|\psi_x\|_2^2 + \rho_2\|\psi_t\|_2^2 + J\psi_t^2(L,t) - \gamma\int_0^L \theta\psi dx + \rho_1\int_0^L \chi_t\omega_t dx. \tag{3.3}$$

Using Young's and Hölder's inequalities, we deduce from (3.1) that

$$\begin{aligned}
\|\omega_t\|_2^2 &= \int_0^L \left(-\int_0^x \psi_t(\lambda,t)d\lambda + \int_0^L \psi_t(\lambda,t)d\lambda\right)^2 dx \\
&\leq 2\int_0^L \left(\int_0^x \psi_t(\lambda,t)d\lambda\right)^2 dx + 2\int_0^L \left(\int_0^L \psi_t(\lambda,t)d\lambda\right)^2 dx \\
&\leq 2\int_0^L \left(\int_0^x 1 d\lambda\right)\left(\int_0^x \psi_t^2(\lambda,t)d\lambda\right) dx + 2\int_0^L \left(\int_0^L 1 d\lambda\right)\left(\int_0^L \psi_t^2(\lambda,t)d\lambda\right) dx \\
&\leq 4L^2\int_0^L \psi_t^2(\lambda,t)d\lambda,
\end{aligned}$$

then

$$\|\omega_t\|_2^2 \leq 4L^2 \|\psi_t\|_2^2. \tag{3.4}$$

Therefore, using Young's inequality and (3.4), for ε_1, $\varepsilon_3 > 0$, we get

$$\rho_1 \int_0^L \chi_t \omega_t dx \leq \rho_1 \varepsilon_1 \left\| \chi_t \right\|_2^2 + \frac{\rho_1 L^2}{\varepsilon_1} \left\| \psi_t \right\|_2^2 , \tag{3.5}$$

and thanks to Young's and Poincaré's inequalities, we have (C_0 is Poincaré's constant)

$$-\gamma \int_0^L \theta \psi dx \leq \gamma \varepsilon_3 \left\| \theta \right\|_2^2 + \frac{\gamma C_0}{4\varepsilon_3} \left\| \psi_x \right\|_2^2 . \tag{3.6}$$

Summing up (3.3)-(3.6), it appears that, for $C_p = \max\{4L^2, C_0\}$,

$$\begin{aligned} U_1'(t) &\leq -b \left\| \psi_x \right\|_2^2 + \rho_2 \left\| \psi_t \right\|_2^2 + J\psi_t^2(L,t) + \gamma\varepsilon_3 \left\| \theta \right\|_2^2 + \frac{\gamma C_p}{4\varepsilon_3} \left\| \psi_x \right\|_2^2 + \rho_1 \left(\varepsilon_1 \left\| \chi_t \right\|_2^2 + \frac{C_p}{4\varepsilon_1} \left\| \psi_t \right\|_2^2 \right) \\ &\leq \left(\frac{\gamma C_p}{4\varepsilon_3} - b \right) \left\| \psi_x \right\|_2^2 + \left(\frac{\rho_1 C_p}{4\varepsilon_1} + \rho_2 \right) \left\| \psi_t \right\|_2^2 + J\psi_t^2(L,t) + \gamma\varepsilon_3 \left\| \theta \right\|_2^2 + \rho_1 \varepsilon_1 \left\| \chi_t \right\|_2^2 \end{aligned}$$

which is exactly what we announced in (3.2).

Let $g(x)$ be a C^1-function satisfying $g(0) = -g(L) = 2$ such as $g(x) = -\frac{4}{L}x + 2$.

Lemma 2. For the functional

$$U_2(t) := \rho_2 \int_0^L \psi_t \psi_x g(x) dx,$$

we have, for any $\varepsilon_2 > 0$,

$$U_2'(t) \leq -\rho_2 \psi_t^2(L,t) - b\psi_x^2(L,t) - b\psi_x^2(0,t) + \left(\frac{2b}{L} + 2k\varepsilon_2 \right) \left\| \psi_x \right\|_2^2 + \frac{k}{2\varepsilon_2} \left\| \chi_x + \psi \right\|_2^2 + \frac{2\rho_2}{L} \left\| \psi_t \right\|_2^2 . \tag{3.7}$$

Proof. Again, a differentiation followed by integration by parts yields

$$\begin{aligned} U_2'(t) &= \rho_2 \int_0^L \psi_t \psi_{xt} g(x) dx + \rho_2 \int_0^L \psi_{tt} \psi_x g(x) dx \\ &= \rho_2 \int_0^L \frac{d\psi_t^2}{2dx} g(x) dx + b \int_0^L \frac{d\psi_x^2}{2dx} g(x) dx - k \int_0^L (\chi_x + \psi) \psi_x g(x) dx \\ &= \frac{\rho_2}{2} \left[\psi_t^2 g(x) \right]_0^L - \frac{\rho_2}{2} \int_0^L \psi_t^2 g'(x) dx + \frac{b}{2} \left[\psi_x^2 g(x) \right]_0^L - \frac{b}{2} \int_0^L \psi_x^2 g'(x) dx - k \int_0^L (\chi_x + \psi) \psi_x g(x) dx, \end{aligned}$$

and therefore, for any $\varepsilon_2 > 0$,

$$\begin{aligned} U_2'(t) &\leq -\rho_2 \psi_t^2(L,t) + \frac{2\rho_2}{L} \left\| \psi_t \right\|_2^2 - b\psi_x^2(L,t) - b\psi_x^2(0,t) + \frac{2b}{L} \left\| \psi_x \right\|_2^2 + 2k\varepsilon_2 \left\| \psi_x \right\|_2^2 + \frac{k}{2\varepsilon_2} \left\| \chi_x + \psi \right\|_2^2 \\ &\leq -\rho_2 \psi_t^2(L,t) - b\psi_x^2(L,t) - b\psi_x^2(0,t) + \left(\frac{2b}{L} + 2k\varepsilon_2 \right) \left\| \psi_x \right\|_2^2 + \frac{k}{2\varepsilon_2} \left\| \chi_x + \psi \right\|_2^2 + \frac{2\rho_2}{L} \left\| \psi_t \right\|_2^2 . \end{aligned}$$

The proof of (3.7) is complete.

Lemma 3. Differentiating the functional

$$U_3(t) := -\rho_1 \int_0^L (\chi_x + \psi) \int_0^x \chi_t(y,t) dy dx$$

and estimating give, for any ε_0, $\varepsilon_5, \varepsilon_6 > 0$,

$$\begin{aligned} U_3'(t) &\leq (\gamma\varepsilon_6 - k) \left\| \chi_x + \psi \right\|_2^2 + \left(\rho_1 + \frac{\rho_1 L}{4} \varepsilon_5 \right) \left\| \chi_t \right\|_2^2 + \frac{\rho_1}{\varepsilon_5} \left\| \psi_t \right\|_2^2 \\ &\quad + \frac{\gamma}{4\varepsilon_6} \left\| \theta \right\|_2^2 + L\rho_1 \varepsilon_0 \left\| \chi_t \right\|_2^2 + \frac{\rho_1}{4\varepsilon_0} \chi_t^2(L). \end{aligned} \tag{3.8}$$

Proof. Clearly

$$
\begin{aligned}
U_3'(t) &= -\rho_1 \int_0^L (\chi_x+\psi)_t \int_0^x \chi_t(y,t)dydx - \rho_1 \int_0^L (\chi_x+\psi) \int_0^x \chi_{tt}(y,t)dydx \\
&= -\rho_1 \int_0^L \chi_{xt} \int_0^x \chi_t(y,t)dydx - \rho_1 \int_0^L \psi_t \int_0^x \chi_t(y,t)dydx \\
&\quad - \int_0^L (\chi_x+\psi) \int_0^x k(\chi_x+\psi)_x (y,t)dydx + \gamma \int_0^L (\chi_x+\psi) \int_0^x \theta_x(y,t)dydx \\
&= \rho_1 \|\chi_t\|_2^2 - \rho_1 \int_0^L \psi_t \int_0^x \chi_t(y,t)dydx - k\|\chi_x+\psi\|_2^2 + \gamma \int_0^L (\chi_x+\psi)\theta dx - \rho_1 \chi_t(L) \int_0^L \chi_t(y,t)dy.
\end{aligned}
$$

Evaluating terms there, we may write, for any $\varepsilon_5, \varepsilon_6 > 0$,

$$
\begin{aligned}
U_3'(t) &= -k\|\chi_x+\psi\|_2^2 + \rho_1 \|\chi_t\|_2^2 - \rho_1 \int_0^L \psi_t \int_0^x \chi_t(y,t)dydx + \gamma \int_0^L (\chi_x+\psi)\theta dx - \rho_1 \chi_t(L) \int_0^L \chi_t(y,t)dy \\
&\le -k\|\chi_x+\psi\|_2^2 + \rho_1 \|\chi_t\|_2^2 + \frac{\rho_1}{\varepsilon_5}\|\psi_t\|_2^2 + \frac{\rho_1 L^2}{4}\varepsilon_5 \|\chi_t\|_2^2 + \gamma\varepsilon_6 \|\chi_x+\psi\|_2^2 \\
&\quad + \frac{\gamma}{4\varepsilon_6}\|\theta\|_2^2 + L\rho_1\varepsilon_{20}\|\chi_t\|_2^2 + \frac{\rho_1}{4\varepsilon_{20}}\chi_t^2(L) \\
&\le (\gamma\varepsilon_6 - k)\|\chi_x+\psi\|_2^2 + \left(\rho_1 + \frac{\rho_1 L}{4}\varepsilon_5\right)\|\chi_t\|_2^2 + \frac{\rho_1}{\varepsilon_5}\|\psi_t\|_2^2 + \frac{\gamma}{4\varepsilon_6}\|\theta\|_2^2 + L\rho_1\varepsilon_0\|\chi_t\|_2^2 + \frac{\rho_1}{4\varepsilon_0}\chi_t^2(L).
\end{aligned}
$$

This finishes the proof of (3.8).

Lemma 4. For the functional

$$
U_4(t) := \tau_0 \rho_3 \int_0^L \theta(x,t) \left(\int_0^x \sigma(y,t)\, dy \right) dx,
$$

it holds that, for any $\varepsilon_0 > 0$,

$$
U_4'(t) \le \left(\tau_0 + \frac{(\delta\rho_3 L)^2 + (\gamma\tau_0)^2}{4\varepsilon_0} \right) \|\sigma\|_2^2 + \varepsilon_0 \|\chi_t\|_2^2 + (\varepsilon_0 - \kappa\rho_3)\|\theta\|_2^2. \tag{3.9}
$$

Proof. In virtue of the last two equations in (2.8), we have

$$
\begin{aligned}
U_4'(t) &= \tau_0\rho_3 \int_0^L \theta_t \left(\int_0^x \sigma(y,t)\, dy \right) dx + \tau_0\rho_3 \int_0^L \theta \left(\int_0^x \sigma_t(y,t)\, dy \right) dx \\
&= -\tau_0 \int_0^L (\sigma_x + \gamma\chi_{tx}) \left(\int_0^x \sigma(y,t)\, dy \right) dx - \rho_3 \int_0^L \theta \left(\int_0^x (\delta\sigma + \kappa\theta_x)(y,t)\, dy \right) dx \\
&= -\tau_0 \int_0^L \sigma_x \left(\int_0^x \sigma(y,t)\, dy \right) dx - \gamma\tau_0 \int_0^L \chi_{tx} \left(\int_0^x \sigma(y,t)\, dy \right) dx \\
&\quad -\delta\rho_3 \int_0^L \theta \left(\int_0^x \sigma(y,t)\, dy \right) dx - \kappa\rho_3 \int_0^L \theta \left(\int_0^x \theta_x(y,t)\, dy \right) dx.
\end{aligned}
$$

Then, integrating by parts, we entail

$$
\begin{aligned}
U_4'(t) &= -\tau_0 \left[\sigma \left(\int_0^x \sigma(y,t)\, dy \right) \right]_0^L + \tau_0 \|\sigma\|_2^2 - \gamma\tau_0 \left[\chi_t \left(\int_0^x \sigma(y,t)\, dy \right) \right]_0^L \\
&\quad + \gamma\tau_0 \int_0^L \chi_t \sigma dx - \delta\rho_3 \int_0^L \theta \left(\int_0^x \sigma(y,t)\, dy \right) dx - \kappa\rho_3 \|\theta\|_2^2 \\
&= \tau_0 \|\sigma\|_2^2 + \gamma\tau_0 \int_0^L \chi_t\sigma dx - \delta\rho_3 \int_0^L \theta \left(\int_0^x \sigma(y,t)\, dy \right) dx - \kappa\rho_3 \|\theta\|_2^2. \qquad (3.10)
\end{aligned}
$$

After, estimating the second and third terms in (3.10), we get

$$
U_4'(t) \le \tau_0 \|\sigma\|_2^2 + \varepsilon_0 \|\chi_t\|_2^2 + \frac{(\gamma\tau_0)^2}{4\varepsilon_0}\|\sigma\|_2^2 + \varepsilon_0 \|\theta\|_2^2 + \frac{(\delta\rho_3)^2}{4\varepsilon_0} L^2 \|\sigma\|_2^2 - \kappa\rho_3 \|\theta\|_2^2
$$

or

$$U_4'(t) \leq \left(\tau_0 + \frac{(\delta\rho_3 L)^2 + (\gamma\tau_0)^2}{4\varepsilon_0}\right) \|\sigma\|_2^2 + \varepsilon_0 \|\chi_t\|_2^2 + (\varepsilon_0 - \kappa\rho_3) \|\theta\|_2^2 .$$

The proof of (3.9) is complete.

We introduce the number

$$\mathcal{D} := \frac{\gamma b}{k} - \frac{1}{\gamma}\left(\frac{\kappa\rho_2}{\tau_0} - b\rho_3\right)\left(1 - \frac{b\rho_1}{k\rho_2}\right). \tag{3.11}$$

Lemma 5. If $\mathcal{D}$ =0, the functional

$$U_5(t) := -\rho_2 \int_0^L \psi_t (\chi_x + \psi) dx - \frac{b\rho_1}{k}\int_0^L \chi_t \psi_x dx + \frac{\rho_2\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2} - 1\right)\int_0^L \theta\psi_t dx + \frac{\rho_2}{\gamma}\left(1 - \frac{b\rho_1}{k\rho_2}\right)\int_0^L \sigma\psi_x dx$$

verifies, for any ε_0, ε_4, $\varepsilon_9 > 0$,

$$\begin{aligned} U_5'(t) \leq\ & k\left(1 + \frac{\rho_3^2 A}{4}\right)\|\chi_x + \psi\|_2^2 - \rho_2\|\psi_t\|_2^2 + b\varepsilon_4 \psi_x^2(L,t) + \frac{b}{\varepsilon_4}(\chi_x + \psi)^2(L,t) + \frac{b\rho_1}{k\varepsilon_0}\chi_t^2(L,t) \\ & + \frac{1}{4}\left(\frac{b\rho_1\varepsilon_0}{k} + \frac{A\rho_2}{\varepsilon_9}\right)\psi_t^2(L,t) + A\rho_2\varepsilon_9\sigma^2(L,t) + kA\|\theta\|_2^2 + \varepsilon_0\frac{A\delta\rho_2}{\tau_0}\|\psi_x\|_2^2 + \frac{A\delta\rho_2}{4\varepsilon_0\tau_0}\|\sigma\|_2^2, \end{aligned} \tag{3.12}$$

where $A := \frac{1}{\gamma}\left|\frac{b\rho_1}{k\rho_2} - 1\right|$.

Proof. First, we differentiate each term in $U_5(t)$ separately with respect to time, take into account the equations in (2.8) and integrate by parts, we get

$$\begin{aligned} -\rho_2\frac{d}{dt}\int_0^L \psi_t(\chi_x+\psi)dx &= -\rho_2\int_0^L \psi_{tt}(\chi_x+\psi)dx - \rho_2\int_0^L \psi_t(\chi_x+\psi)_t dx \\ &= -\int_0^L (\chi_x+\psi)[b\psi_{xx} - k(\chi_x+\psi)]dx - \rho_2\int_0^L \psi_t(\chi_{xt}+\psi_t)dx \\ &= -b\psi_x(L,t)(\chi_x+\psi)(L,t) + b\int_0^L (\chi_x+\psi)_x\psi_x dx \\ &\quad + k\|\chi_x+\psi\|_2^2 - \rho_2\int_0^L \psi_t\chi_{xt}dx - \rho_2\|\psi_t\|_2^2, \end{aligned} \tag{3.13, 3.14}$$

$$\begin{aligned} -\frac{b\rho_1}{k}\frac{d}{dt}\int_0^L \chi_t\psi_x dx &= -\frac{b\rho_1}{k}\int_0^L \chi_{tt}\psi_x dx - \frac{b\rho_1}{k}\int_0^L \chi_t\psi_{xt}dx \\ &= -\frac{b}{k}\int_0^L \psi_x[k(\chi_x+\psi)_x - \gamma\theta_x]dx - \frac{b\rho_1}{k}\int_0^L \chi_t\psi_{xt}dx \\ &= -b\int_0^L \psi_x(\chi_x+\psi)_x dx + \frac{\gamma b}{k}\int_0^L \psi_x\theta_x dx - \frac{b\rho_1}{k}\int_0^L \chi_t\psi_{xt}dx, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \frac{\rho_2\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\frac{d}{dt}\int_0^L \theta\psi_t dx &= \frac{\rho_2\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \theta_t\psi_t dx + \frac{\rho_2\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \theta\psi_{tt}dx \\ &= -\frac{\rho_2}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \psi_t(\sigma_x + \gamma\chi_{tx})dx \\ &\quad + \frac{\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \theta[b\psi_{xx} - k(\chi_x+\psi)]dx \\ &= -\frac{\rho_2}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \psi_t\sigma_x dx - \rho_2\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \psi_t\chi_{tx}dx \\ &\quad - \frac{b\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \theta_x\psi_x dx - \frac{k\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L \theta(\chi_x+\psi)dx \end{aligned} \tag{3.16}$$

and

$$
\begin{aligned}
\frac{\rho_2}{\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\frac{d}{dt}\int_0^L \sigma\psi_x dx &= \frac{\rho_2}{\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \sigma_t\psi_x dx+\frac{\rho_2}{\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \sigma\psi_{xt}dx \\
&= -\frac{\rho_2}{\tau_0\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \psi_x\left(\delta\sigma+\kappa\theta_x\right)dx+\frac{\rho_2}{\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \sigma\psi_{xt}dx \\
&= -\frac{\delta\rho_2}{\tau_0\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \psi_x\sigma dx-\frac{\kappa\rho_2}{\tau_0\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \psi_x\theta_x dx \\
&\quad +\frac{\rho_2}{\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\sigma(L,t)\psi_t(L,t)-\frac{\rho_2}{\gamma}\left(1-\frac{b\rho_1}{k\rho_2}\right)\int_0^L \sigma_x\psi_t dx. \qquad (3.17)
\end{aligned}
$$

Next, we sum up the previous expressions (3.14)-(3.17), we find

$$
\begin{aligned}
U_5'(t) &= -b\psi_x(L,t)\left(\chi_x+\psi\right)(L,t)+b\int_0^L\left(\chi_x+\psi\right)_x\psi_x dx+k\left\|\chi_x+\psi\right\|_2^2-\rho_2\int_0^L\psi_t\chi_{xt}dx \\
&\quad -\rho_2\left\|\psi_t\right\|_2^2-b\int_0^L\psi_x\left(\chi_x+\psi\right)_x dx+\frac{\gamma b}{k}\int_0^L\psi_x\theta_x dx-\frac{b\rho_1}{k}\chi_t(L,t)\psi_t(L,t)+\frac{b\rho_1}{k}\int_0^L\chi_{tx}\psi_t dx \\
&\quad -\frac{\rho_2}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\psi_t\sigma_x dx-\rho_2\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\psi_t\chi_{tx}dx-\frac{b\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\theta_x\psi_x dx \\
&\quad -\frac{k\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\theta\left(\chi_x+\psi\right)dx+\frac{\delta\rho_2}{\tau_0\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\psi_x\sigma dx+\frac{\kappa\rho_2}{\tau_0\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\psi_x\theta_x dx \\
&\quad -\frac{\rho_2}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\sigma(L,t)\psi_t(L,t)+\frac{\rho_2}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\sigma_x\psi_t dx.
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
U_5'(t) &= k\left\|\chi_x+\psi\right\|_2^2-\rho_2\left\|\psi_t\right\|_2^2-b\psi_x(L,t)\left(\chi_x+\psi\right)(L,t)-\frac{b\rho_1}{k}\chi_t(L,t)\psi_t(L,t) \\
&\quad -\frac{\rho_2}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\sigma(L,t)\psi_t(L,t)-\frac{k\rho_3}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\theta\left(\chi_x+\psi\right)dx \\
&\quad +\frac{\delta\rho_2}{\tau_0\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\psi_x\sigma dx+\left[\frac{\gamma b}{k}-\frac{1}{\gamma}\left(\frac{\kappa\rho_2}{\tau_0}-b\rho_3\right)\left(1-\frac{b\rho_1}{k\rho_2}\right)\right]\int_0^L\theta_x\psi_x dx. \qquad (3.18)
\end{aligned}
$$

We can estimate terms in the right side of (3.18) as follows, using Young's inequality, for any $\varepsilon_0,\varepsilon_4,\varepsilon_9>0$:

$$
\frac{\delta\rho_2}{\gamma\tau_0}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\psi_x\sigma dx\le\frac{A\delta\rho_2}{\tau_0}\left(\varepsilon_0\left\|\psi_x\right\|_2^2+\frac{1}{4\varepsilon_0}\left\|\sigma\right\|_2^2\right), \qquad (3.19)
$$

$$
-k\rho_3\frac{1}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\int_0^L\theta\left(\chi_x+\psi\right)dx\le kA\left(\left\|\theta\right\|_2^2+\frac{\rho_3^2}{4}\left\|\chi_x+\psi\right\|_2^2\right), \qquad (3.20)
$$

$$
-b\psi_x(L,t)\left(\chi_x+\psi\right)(L,t)\le b\left(\varepsilon_4\psi_x^2(L,t)+\frac{1}{\varepsilon_4}\left(\chi_x+\psi\right)^2(L,t)\right), \qquad (3.21)
$$

$$
-\frac{b\rho_1}{k}\chi_t(L,t)\psi_t(L,t)\le\frac{b\rho_1}{k}\left(\frac{1}{\varepsilon_0}\chi_t^2(L,t)+\frac{\varepsilon_0}{4}\psi_t^2(L,t)\right) \qquad (3.22)
$$

and

$$
-\rho_2\frac{1}{\gamma}\left(\frac{b\rho_1}{k\rho_2}-1\right)\sigma(L,t)\psi_t(L,t)\le A\rho_2\left(\varepsilon_9\sigma^2(L,t)+\frac{1}{4\varepsilon_9}\psi_t^2(L,t)\right). \qquad (3.23)
$$

Gathering (3.19)-(3.23) and using the fact that $\mathcal{D}=0$, we infer from (3.18), that

$$\begin{aligned} U_5'(t) &\leq k\left\|\chi_x+\psi\right\|_2^2-\rho_2\left\|\psi_t\right\|_2^2+b\left(\varepsilon_4\psi_x^2(L,t)+\frac{1}{\varepsilon_4}\left(\chi_x+\psi\right)^2(L,t)\right) \\ &+\frac{b\rho_1}{k}\left(\varepsilon_9\chi_t^2(L,t)+\frac{1}{4\varepsilon_9}\psi_t^2(L,t)\right)+A\rho_2\left(\varepsilon_9\sigma^2(L,t)+\frac{1}{4\varepsilon_9}\psi_t^2(L,t)\right) \\ &+kA\left(\|\theta\|_2^2+\frac{\rho_3^2}{4}\left\|\chi_x+\psi\right\|_2^2\right)+\frac{A\delta\rho_2}{\tau_0}\left(\varepsilon_0\left\|\psi_x\right\|_2^2+\frac{1}{4\varepsilon_0}\|\sigma\|_2^2\right) \end{aligned}$$

or

$$\begin{aligned} U_5'(t) \leq\ & k\left(1+\frac{\rho_3^2A}{4}\right)\left\|\chi_x+\psi\right\|_2^2-\rho_2\left\|\psi_t\right\|_2^2+b\varepsilon_4\psi_x^2(L,t)+\frac{b}{\varepsilon_4}\left(\chi_x+\psi\right)^2(L,t)+\frac{b\rho_1}{k}\varepsilon_9\chi_t^2(L,t) \\ &+\frac{1}{4}\left(\frac{b\rho_1}{\varepsilon_9 k}+\frac{A\rho_2}{\varepsilon_9}\right)\psi_t^2(L,t)+A\rho_2\varepsilon_9\sigma^2(L,t)+kA\|\theta\|_2^2+\varepsilon_0\frac{A\delta\rho_2}{\tau_0}\left\|\psi_x\right\|_2^2+\frac{A\delta\rho_2}{4\varepsilon_0\tau_0}\|\sigma\|_2^2. \end{aligned}$$

This finishes the proof of (3.12).

Lemma 6. The rate of change of

$$U_6(t):=m\chi_t(0,t)\chi(0,t)$$

satisfies, for any $\varepsilon_0>0$,

$$U_6'(t)\leq\left(m+\frac{K^2}{4\varepsilon_0}\right)\chi_t^2(0,t)+\left(\varepsilon_0-1\right)\chi^2(0,t). \tag{3.24}$$

Proof. Indeed, from the first equation in (2.8), it holds that

$$U_6'(t)=m\chi_t^2(0,t)+m\chi_{tt}(0,t)\chi(0,t)=m\chi_t^2(0,t)+(\tau(t)+\mu\chi_t(L,t))\chi(0,t).$$

Then, by our suggested control,

$$\begin{aligned} U_6'(t) &= m\chi_t^2(0,t)+\left[-K\chi_t(0,t)-\chi(0,t)\right]\chi(0,t) \\ &= m\chi_t^2(0,t)-K\chi_t(0,t)\chi(0,t)-\chi^2(0,t) \\ &\leq m\chi_t^2(0,t)-\chi^2(0,t)+\varepsilon_0\chi^2(0,t)+\frac{K^2}{4\varepsilon_0}\chi_t^2(0,t) \\ &\leq \left(m+\frac{K^2}{4\varepsilon_0}\right)\chi_t^2(0,t)+\left(\varepsilon_0-1\right)\chi^2(0,t). \end{aligned}$$

The proof of (3.24) is complete.

Lemma 7. The functional

$$U_7(t):=\rho_1\int_0^L\chi_t\left(\chi_x+\psi\right)g(x)dx-\frac{\gamma\tau_0}{\kappa}\int_0^L\sigma\left(\chi_x+\psi\right)g(x)dx-\frac{\tau_0\rho_3}{\kappa}\int_0^L\sigma\theta g(x)dx,$$

where $g(x)=-\frac{4}{L}x+2$, fulfills, for any ε_0, $\varepsilon_7>0$,

$$\begin{aligned} U_7'(t) \leq\ & -k\left(\chi_x+\psi\right)^2(L,t)+\left(\varepsilon_0+\frac{2k}{L}\right)\left\|\chi_x+\psi\right\|_2^2-\rho_1\chi_t^2(L,t) \\ &-k\rho_1\chi_t^2(0,t)+\rho_1\left(\varepsilon_7+\frac{2}{L}\right)\left\|\chi_t\right\|_2^2+\left(\varepsilon_0+\frac{2\rho_3}{L}\right)\|\theta\|_2^2-\frac{\tau_0}{\kappa}\sigma^2(L,t)-\frac{\tau_0}{\kappa}\sigma^2(0,t) \\ &+\left(\varepsilon_0+\frac{\rho_1}{\varepsilon_7}\right)\left\|\psi_t\right\|_2^2+\left[\frac{2\tau_0}{L\kappa}+\left[(\gamma\delta)^2+(\gamma\tau_0)^2+(\delta\rho_3)^2\right]\frac{1}{\varepsilon_0\kappa^2}\right]\|\sigma\|_2^2. \end{aligned} \tag{3.25}$$

Proof. A simple differentiation taking into account the equations in (2.8), yields

$$\begin{aligned}
U_7'(t) &= \rho_1\int_0^L \chi_{tt}\left(\chi_x+\psi\right)g(x)dx+\rho_1\int_0^L \chi_t\left(\chi_x+\psi\right)_t g(x)dx-\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma_t\left(\chi_x+\psi\right)g(x)dx\\
&\quad -\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\left(\chi_x+\psi\right)_t g(x)dx-\frac{\tau_0\rho_3}{\kappa}\int_0^L \sigma_t\theta g(x)dx-\frac{\tau_0\rho_3}{\kappa}\int_0^L \sigma\theta_t g(x)dx\\
&= \int_0^L \left[k\left(\chi_x+\psi\right)_x-\gamma\theta_x\right]\left(\chi_x+\psi\right)g(x)dx+\rho_1\int_0^L \chi_t\left(\chi_x+\psi\right)_t g(x)dx\\
&\quad +\frac{\gamma}{\kappa}\int_0^L \left(\delta\sigma+\kappa\theta_x\right)\left(\chi_x+\psi\right)g(x)dx-\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\left(\chi_x+\psi\right)_t g(x)dx\\
&\quad +\frac{\rho_3}{\kappa}\int_0^L \left(\delta\sigma+\kappa\theta_x\right)\theta g(x)dx-\frac{\tau_0\rho_3}{\kappa}\int_0^L \sigma\theta_t g(x)dx
\end{aligned}$$

or

$$\begin{aligned}
U_7'(t) &= \int_0^L k\left(\chi_x+\psi\right)_x\left(\chi_x+\psi\right)g(x)dx-\gamma\int_0^L \theta_x\left(\chi_x+\psi\right)g(x)dx\\
&\quad +\rho_1\int_0^L \chi_t\left(\chi_x+\psi\right)_t g(x)dx+\frac{\gamma\delta}{\kappa}\int_0^L \sigma\left(\chi_x+\psi\right)g(x)dx+\gamma\int_0^L \theta_x\left(\chi_x+\psi\right)g(x)dx\\
&\quad -\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\chi_{xt}g(x)dx-\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\psi_t g(x)dx+\frac{\rho_3\delta}{\kappa}\int_0^L \sigma\theta g(x)dx\\
&\quad +\rho_3\int_0^L \theta_x\theta g(x)dx+\frac{\tau_0}{\kappa}\int_0^L \sigma\sigma_x g(x)dx+\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\chi_{tx}g(x)dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
U_7'(t) &= k\int_0^L \frac{d}{2dx}\left(\chi_x+\psi\right)^2 g(x)dx+\rho_1\int_0^L \frac{d\chi_t^2}{2dx}g(x)dx+\rho_1\int_0^L \chi_t\psi_t g(x)dx\\
&\quad +\frac{\gamma\delta}{\kappa}\int_0^L \sigma\left(\chi_x+\psi\right)g(x)dx-\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\psi_t g(x)dx+\frac{\rho_3\delta}{\kappa}\int_0^L \sigma\theta g(x)dx\\
&\quad +\rho_3\int_0^L \frac{d\theta^2}{2dx}g(x)dx+\frac{\tau_0}{\kappa}\int_0^L \frac{d\sigma^2}{2dx}g(x)dx.
\end{aligned}$$

Next, we integrate by parts, we get

$$\begin{aligned}
U_7'(t) &= \frac{k}{2}\left[\left(\chi_x+\psi\right)^2 g(x)\right]_0^L-\frac{k}{2}\int_0^L \left(\chi_x+\psi\right)^2 g'(x)dx+\frac{\rho_1}{2}\left[\chi_t^2 g(x)\right]_0^L\\
&\quad -\frac{\rho_1}{2}\int_0^L \chi_t^2 g'(x)dx+\rho_1\int_0^L \chi_t\psi_t g(x)dx+\frac{\gamma\delta}{\kappa}\int_0^L \sigma\left(\chi_x+\psi\right)g(x)dx-\frac{\gamma\tau_0}{\kappa}\int_0^L \sigma\psi_t g(x)dx\\
&\quad +\frac{\rho_3\delta}{\kappa}\int_0^L \sigma\theta g(x)dx+\frac{\rho_3}{2}\left[\theta^2 g(x)\right]_0^L-\frac{\rho_3}{2}\int_0^L \theta^2 g'(x)dx+\frac{\tau_0}{2\kappa}\left[\sigma^2 g(x)\right]_0^L-\frac{\tau_0}{2\kappa}\int_0^L \sigma^2 g'(x)dx.
\end{aligned}$$

By Young's inequality, we may write, for any $\varepsilon_0, \varepsilon_7 > 0$,

$$\begin{aligned}
U_7'(t) &\leq -k\left(\chi_x+\psi\right)^2(L,t)+\frac{2k}{L}\left\|\chi_x+\psi\right\|_2^2-\rho_1\chi_t^2(L,t)\\
&\quad -k\rho_1\chi_t^2(0,t)+\frac{2\rho_1}{L}\left\|\chi_t\right\|_2^2+\varepsilon_7\rho_1\left\|\chi_t\right\|_2^2+\frac{\rho_1}{\varepsilon_7}\left\|\psi_t\right\|_2^2+\varepsilon_0\left\|\chi_x+\psi\right\|_2^2\\
&\quad +\frac{1}{\varepsilon_0}\left(\frac{\gamma\delta}{\kappa}\right)^2\left\|\sigma\right\|_2^2+\varepsilon_0\left\|\psi_t\right\|_2^2+\frac{1}{\varepsilon_0}\left(\frac{\gamma\tau_0}{\kappa}\right)^2\left\|\sigma\right\|_2^2+\varepsilon_0\left\|\theta\right\|_2^2+\frac{1}{\varepsilon_0}\left(\frac{\delta\rho_3}{\kappa}\right)^2\left\|\sigma\right\|_2^2\\
&\quad +\frac{2\rho_3}{L}\left\|\theta\right\|_2^2-\frac{\tau_0}{\kappa}\sigma^2(L,t)-\frac{\tau_0}{\kappa}\sigma^2(0,t)+\frac{2\tau_0}{L\kappa}\left\|\sigma\right\|_2^2. \qquad (3.26)
\end{aligned}$$

Simplifying the expression (3.26), we end up with

$$\begin{aligned} U_7'(t) &\le -k\left(\chi_x+\psi\right)^2(L,t)+\left(\varepsilon_0+\frac{2k}{L}\right)\|\chi_x+\psi\|_2^2-\rho_1\chi_t^2(L,t)-k\rho_1\chi_t^2(0,t) \\ &\quad +\rho_1\left(\varepsilon_7+\frac{2}{L}\right)\|\chi_t\|_2^2+\left(\varepsilon_0+\frac{2\rho_3}{L}\right)\|\theta\|_2^2-\frac{\tau_0}{\kappa}\sigma^2(L,t)-\frac{\tau_0}{\kappa}\sigma^2(0,t) \\ &\quad +\left(\varepsilon_0+\frac{\rho_1}{\varepsilon_7}\right)\|\psi_t\|_2^2+\left[\frac{2\tau_0}{L\kappa}+\left[(\gamma\delta)^2+(\gamma\tau_0)^2+(\delta\rho_3)^2\right]\frac{1}{\varepsilon_0\kappa^2}\right]\|\sigma\|_2^2 . \end{aligned}$$

This is what we wanted to prove in (3.25).

Lemma 8. The rate of change of the functional

$$U_8(t)=-\rho_1\rho_3\int_0^L\theta\int_0^x\chi_t(y,t)dydx$$

fulfills, for any ε_0, ε_8, $\varepsilon_{21}>0$,

$$\begin{aligned} U_8'(t) &\le \rho_1\left(\varepsilon_0+L(\varepsilon_{21}+\gamma\varepsilon_0)-\gamma\right)\|\chi_t\|_2^2+\rho_3\left(\gamma+k\varepsilon_8\right)\|\theta\|_2^2+\frac{k\rho_3}{4\varepsilon_8}\|\chi_x+\psi\|_2^2 \\ &\quad +\frac{\rho_1}{4\varepsilon_0}\|\sigma\|_2^2+\frac{\rho_1}{4\varepsilon_{21}}\sigma^2(L)+\frac{\gamma\rho_1}{4\varepsilon_0}\chi_t^2(L). \end{aligned} \tag{3.27}$$

Proof. It is easy to see that

$$\begin{aligned} U_8'(t) &= -\rho_3\int_0^L\theta\int_0^x\left[k\left(\chi_x+\psi\right)_x-\gamma\theta_x\right](y,t)dydx+\rho_1\int_0^L\left[\sigma_x+\gamma\chi_{tx}\right]\int_0^x\chi_t(y,t)dydx \\ &= -\rho_3\int_0^L\theta\left[k\left(\chi_x+\psi\right)-\gamma\theta\right]dx+\rho_1\int_0^L\sigma_x\int_0^x\chi_t(y,t)dydx+\gamma\rho_1\int_0^L\chi_{tx}\int_0^x\chi_t(y,t)dydx \\ &= -k\rho_3\int_0^L\theta\left(\chi_x+\psi\right)dx+\gamma\rho_3\|\theta\|_2^2-\rho_1\int_0^L\sigma\chi_tdx+\rho_1\left[\sigma\int_0^x\chi_t(y,t)dy\right]_0^L \\ &\quad +\gamma\rho_1\left[\chi_t\int_0^x\chi_t(y,t)dy\right]_0^L-\gamma\rho_1\|\chi_t\|_2^2 \\ &= -k\rho_3\int_0^L\theta\left(\chi_x+\psi\right)dx+\gamma\rho_3\|\theta\|_2^2-\rho_1\int_0^L\sigma\chi_tdx \\ &\quad -\gamma\rho_1\|\chi_t\|_2^2+\rho_1(\sigma(L)+\gamma\chi_t(L))\int_0^L\chi_t(y,t)dy. \end{aligned}$$

Therefore

$$\begin{aligned} U_8'(t) &\le -\gamma\rho_1\|\chi_t\|_2^2+\gamma\rho_3\|\theta\|_2^2+k\rho_3\varepsilon_8\|\theta\|_2^2+\frac{k\rho_3}{4\varepsilon_8}\|\chi_x+\psi\|_2^2 \\ &\quad +\rho_1\varepsilon_0\|\chi_t\|_2^2+\frac{\rho_1}{4\varepsilon_0}\|\sigma\|_2^2+L\rho_1(\varepsilon_{21}+\gamma\varepsilon_{22})\|\chi_t\|_2^2+\frac{\rho_1}{4\varepsilon_{21}}\sigma^2(L)+\frac{\gamma\rho_1}{4\varepsilon_{22}}\chi_t^2(L) \\ &\le \rho_1\left(\varepsilon_0+L(\varepsilon_{21}+\gamma\varepsilon_0)-\gamma\right)\|\chi_t\|_2^2+\rho_3\left(\gamma+k\varepsilon_8\right)\|\theta\|_2^2 \\ &\quad +\frac{k\rho_3}{4\varepsilon_8}\|\chi_x+\psi\|_2^2+\frac{\rho_1}{4\varepsilon_0}\|\sigma\|_2^2+\frac{\rho_1}{4\varepsilon_{21}}\sigma^2(L)+\frac{\gamma\rho_1}{4\varepsilon_0}\chi_t^2(L). \end{aligned}$$

The proof of (3.27) is complete.

Having introduced all our functionals, we define now the functional V_1 as follows:

$$V_1:=ME_1+\sum_{i=1}^{8}M_iU_i, \tag{3.28}$$

where M, M_i, $i=1,\cdots,8$, are positive constants to be selected later. We can check that V_1 is equivalent to the energy functional E_1 if M is large enough; that is, for some $\beta_1,\beta_2>0$,

$$\beta_1E_1\le V_1\le\beta_2E_1. \tag{3.29}$$

Theorem 2. Assume that $\mathcal{D}=0$ holds. Then, for any $\Psi_0 \in \mathcal{H}$, there exist two positive constants α and β (depending continuously on $E_1(0)$) such that

$$E_1(t) \leq \beta e^{-\alpha t}. \tag{3.30}$$

Proof. Gathering all our previous findings (2.22), (3.2), (3.7)-(3.12), (3.24), (3.25) and (3.27), we obtain

$$\begin{aligned} V_1'(t) \leq\ & A_1\chi_t^2(0,t) + A_2\|\sigma\|_2^2 + A_3\|\psi_x\|_2^2 + A_4\|\psi_t\|_2^2 + A_5\|\chi_t\|_2^2 + A_6\|\theta\|_2^2 + A_7\|\chi_x+\psi\|_2^2 \\ & + A_8\psi_t^2(L,t) + A_9\psi_x^2(L,t) + A_{10}\chi_t^2(L,t) + A_{11}\sigma^2(L,t) + A_{12}(\chi_x+\psi)^2(L,t) \\ & + \left[2M_2(\varepsilon_0-1) + M_6(\varepsilon_0-1)\right]\chi^2(0,t) - \frac{\tau_0}{\kappa}M_7\sigma^2(0,t) - bM_2\psi_x^2(0,t), \end{aligned} \tag{3.31}$$

where

$$A_1 := M_6\left(m + \frac{K^2}{4\varepsilon_0}\right) - k\rho_1 M_7 - MK,$$

$$\begin{aligned} A_2 :=\ & M_4\left(\tau_0 + \frac{(\delta\rho_3 L)^2 + (\gamma\tau_0)^2}{4\varepsilon_0}\right) + M_5\frac{A\delta\rho_2}{4\varepsilon_0\tau_0} + M_8\frac{\rho_1}{4\varepsilon_0} \\ & + M_7\left[\frac{2\tau_0}{L\kappa} + \left[(\gamma\delta)^2 + (\gamma\tau_0)^2 + (\delta\rho_3)^2\right]\frac{1}{\varepsilon_0\kappa^2}\right] - M\frac{\delta}{\kappa}, \end{aligned}$$

$$A_3 := M_1\left(\frac{\gamma C_p}{4\varepsilon_3} - b\right) + M_5\varepsilon_0\frac{A\delta\rho_2}{\tau_0} + M_2\left(\frac{2b}{L} + 2k\varepsilon_2\right),$$

$$A_4 := M_1\left(\frac{\rho_1 C_p}{4\varepsilon_1} + \rho_2\right) + M_3\frac{\rho_1}{\varepsilon_5} + M_2\frac{2\rho_2}{L} + M_7\left(\varepsilon_0 + \frac{k\rho_1}{\varepsilon_7}\right) - M_5\rho_2,$$

$$\begin{aligned} A_5 :=\ & M_1\rho_1\varepsilon_1 + M_3\left(\rho_1(1+L\varepsilon_0) + \frac{\rho_1 L}{4}\varepsilon_5\right) + M_4\varepsilon_0 \\ & + M_7 k\rho_1\left(\varepsilon_7 + \frac{2}{L}\right) + M_8\rho_1\left(\varepsilon_0 + L(\varepsilon_{21} + \gamma\varepsilon_0) - \gamma\right), \end{aligned}$$

$$A_6 := M_1\gamma\varepsilon_3 + M_3\frac{\gamma}{4\varepsilon_6} + M_4(\varepsilon_0 - \kappa\rho_3) + M_7\left(\varepsilon_0 + \frac{2\rho_3}{L}\right) + M_8\rho_3(\gamma + k\varepsilon_8) + M_5 kA,$$

$$A_7 := M_3(\gamma\varepsilon_6 - k) + M_7\left(\varepsilon_0 + \frac{2k}{L}\right) + M_2\frac{k}{2\varepsilon_2} + M_8\frac{k\rho_3}{4\varepsilon_8} + kM_5\left(1 + \frac{\rho_3^2 A}{4}\right),$$

$$A_8 := M_1 J + \frac{1}{4}\left(\frac{b\rho_1\varepsilon_0}{k} + \frac{A\rho_2}{\varepsilon_9}\right)M_5 - \rho_2 M_2,$$

$$A_9 := M_5 b\varepsilon_4 - bM_2,$$

$$A_{10} := \frac{\rho_1}{4\varepsilon_0}M_3 + \frac{b\rho_1}{k\varepsilon_0}M_5 - \rho_1 M_7 + \frac{\gamma\rho_1}{4\varepsilon_0}M_8 - \mu M,$$

$$A_{11} := M_5 A\rho_2\varepsilon_9 - \frac{\tau_0}{\kappa}M_7 + \frac{\rho_1}{4\varepsilon_{21}}M_8$$

and

$$A_{12} := \frac{b}{\varepsilon_4}M_5 - M_7 k.$$

Notice, first, that the three terms A_1, A_2 and A_{10} may be made negative by taking a large M. This gives us the freedom to allow all other terms in these expressions to be as big as we wish. In this respect, ε_0 may be small and ignored at this stage. Consequently, we want to choose the different parameters so as to have

$$M_5\frac{b}{2} + M_2\left(\frac{2b}{L} + 2k\varepsilon_2\right) < M_1\left(b - \frac{\gamma C_p}{4\varepsilon_3}\right),$$

$$M_1\left(\frac{\rho_1 C_p}{4\varepsilon_1} + \rho_2\right) + M_3\frac{\rho_1}{\varepsilon_5} + M_2\frac{2\rho_2}{L} + M_7\frac{k\rho_1}{\varepsilon_7} < M_5\rho_2,$$

$$M_1\rho_1\varepsilon_1 + M_3\left(\rho_1 + \frac{\rho_1 L}{4}\varepsilon_5\right) + M_7 k\rho_1\left(\varepsilon_7 + \frac{2}{L}\right) + L\rho_1\varepsilon_{21}M_8 < M_8\rho_1\gamma,$$

$$M_1\gamma\varepsilon_3 + M_3\frac{\gamma}{4\varepsilon_6} + M_7\frac{2\rho_3}{L} + M_8\rho_3\left(\gamma + k\varepsilon_8\right) + M_5 k\rho_3 A\varepsilon_4 < M_4\kappa\rho_3,$$
$$M_7\frac{2k}{L} + M_2\frac{k}{2\varepsilon_2} + M_8\frac{k\rho_3}{4\varepsilon_8} + kM_5\left(1 + \frac{\rho_3^2 A}{4}\right) < M_3\left(k - \gamma\varepsilon_6\right),$$
$$M_1 J + \frac{A\rho_2}{4\varepsilon_9}M_5 < \rho_2 M_2,$$
$$M_5 b\varepsilon_4 < bM_2,$$
$$M_5 A\rho_2\varepsilon_9 + \frac{\rho_1 M_3}{4\varepsilon_{21}} < \frac{\tau_0}{\kappa}M_7$$

and

$$\frac{b}{\varepsilon_4}M_5 < M_7 k.$$

As we can pick M_4 large, ε_3, ε_4 and ε_8 also may be taken large and ε_6 small. We get

$$M_5\frac{b}{2} + M_2\left(\frac{2b}{L} + 2k\varepsilon_2\right) < M_1 b,$$
$$M_1\left(\frac{\rho_1 C_p}{4\varepsilon_1} + \rho_2\right) + M_3\frac{\rho_1}{\varepsilon_5} + M_2\frac{2\rho_2}{L} + M_7\frac{k\rho_1}{\varepsilon_7} < M_5\rho_2,$$
$$M_1\varepsilon_1 + M_3\left(1 + \frac{L}{4}\varepsilon_5\right) + M_7 k\left(\varepsilon_7 + \frac{2}{L}\right) + L\varepsilon_{21}M_8 < M_8\gamma,$$
$$M_7\frac{2k}{L} + M_2\frac{k}{2\varepsilon_2} + kM_5\left(1 + \frac{\rho_3^2 A}{4}\right) < M_3 k,$$
$$M_1 J + \frac{A\rho_2}{4\varepsilon_9}M_5 < \rho_2 M_2,$$
$$M_5\varepsilon_4 < M_2,$$
$$M_5 A\rho_2\varepsilon_9 + \frac{\rho_1 M_3}{4\varepsilon_{21}} < \frac{\tau_0}{\kappa}M_7$$

and

$$\frac{b}{\varepsilon_4}M_5 + \frac{\rho_1 M_8}{4\varepsilon_{21}} < M_7 k.$$

Next, by taking $\varepsilon_{21} = \frac{\gamma}{4L}$, it is clear that M_8 may be arbitrarily large. This allows ε_1, ε_5 and ε_7 to be large. We are left with

$$M_5\frac{b}{2} + M_2\left(\frac{2b}{L} + 2k\varepsilon_2\right) < M_1 b,$$
$$M_1\rho_2 + M_2\frac{2\rho_2}{L} < M_5\rho_2,$$
$$M_7\frac{2k}{L} + M_2\frac{k}{2\varepsilon_2} + kM_5\left(1 + \frac{\rho_3^2 A}{4}\right) < M_3 k,$$
$$M_1 J + \frac{A\rho_2}{4\varepsilon_9}M_5 < \rho_2 M_2,$$
$$M_5\varepsilon_4 < M_2,$$
$$M_5 A\rho_2\varepsilon_9 < \frac{\tau_0}{\kappa}M_7$$

and

$$\frac{b}{\varepsilon_4}M_5 < M_7 k.$$

Now, it is the turn of M_3 with small ε_2, thus

$$\frac{M_5}{2} + \frac{2}{L}M_2 < M_1,$$
$$M_1 + M_2\frac{2}{L} < M_5,$$
$$M_1 J + \frac{A\rho_2}{4\varepsilon_9}M_5 < \rho_2 M_2,$$

$$M_5\varepsilon_4 < M_2,$$

$$\frac{b}{k}\varepsilon_9 M_5 < kM_7,$$

$$M_5 A\rho_2\varepsilon_9 < \frac{\tau_0}{\kappa}M_7$$

and

$$\frac{b}{\varepsilon_4}M_5 < M_7 k.$$

Clearly we can choose M_7 large allowing ε_9 to be large and ε_4 small, remains

$$\frac{M_5}{2} + \frac{2}{L}M_2 < M_1 \quad \text{and} \quad M_1 + M_2\frac{2}{L} < M_5.$$

This is possible for sufficiently small M_2. Finally, we go back in the reverse order to select the parameters, and then, using (3.31) and the right inequality in (3.29), we get, for some $\tilde{\alpha} > 0$,

$$V_1' \leq -\tilde{\alpha}E_1 \leq -\frac{\tilde{\alpha}}{\beta_2}V_1 := -\alpha V_1. \tag{3.32}$$

By integrating the differential inequality (3.32), we deduce that

$$V_1(t) \leq V_1(0)e^{-\alpha t},$$

which implies (3.30) with $\beta = \frac{1}{\beta_1}V_1(0)$ thanks to the left inequality in (3.29).

4. t^{-1}-stability

In this section we prove that, in the absence of the condition $\mathcal{D} = 0$, we obtain a t^{-1}-stability result for strong solutions of the system. Differentiating the last four equations in (2.8) with respect to time, multiplying by χ_{tt}, ψ_{tt}, θ_t and σ_t, respectively and integrating over $[0, L]$, we find that the second-order energy functional

$$\begin{aligned} E_2(t) := \quad & \frac{1}{2}\left[m_E\chi_{tt}^2(L,t) + J\psi_{tt}^2(L,t) + \rho_1\|\chi_{tt}\|_2^2\right] \\ & + \frac{1}{2}\left[\rho_2\|\psi_{tt}\|_2^2 + \rho_3\|\theta_t\|_2^2 + b\|\psi_{xt}\|_2^2 + k\|(\chi_x+\psi)_t\|_2^2 + \frac{\tau_0}{\kappa}\|\sigma_t\|_2^2\right] \end{aligned} \tag{4.1}$$

satisfies

$$E_2'(t) \leq -\frac{\delta}{\kappa}\|\sigma_t\|_2^2. \tag{4.2}$$

Passing to this second-order energy functional (4.1) allows us to deal with higher order terms. In particular, there is no need to impose the condition $\mathcal{D} = 0$ in order to cancel the last term in (3.18). Indeed, for any $\varepsilon_{10} > 0$, the evaluation

$$\left|\int_0^L \psi_x\theta_x dx\right| \leq \varepsilon_{10}\|\psi_x\|_2^2 + \frac{1}{4\varepsilon_{10}}\|\theta_x\|_2^2 \leq \varepsilon_{10}\|\psi_x\|_2^2 + \frac{1}{2\kappa^2\varepsilon_{10}}\left(\tau_0^2\|\sigma_t\|_2^2 + \delta^2\|\sigma\|_2^2\right)$$

deals comfortably with this term. Moreover, the dissipation property of the system through the thermal effect (which may be seen from the presence of $-\|\sigma\|_2^2$ and $-\|\sigma_t\|_2^2$ in the derivatives (2.22) and (4.2) of the first and second-order energy, respectively), shows that $\varepsilon_{10}\|\psi_x\|_2^2$ may be considered arbitrarily small.

Theorem 3. If $\mathcal{D} \neq 0$, then, for any $\Psi_0 \in D(\mathcal{B})$, there exists a positive constant C (depending continuously on $E_1(0)$ and $E_2(0)$) such that

$$E_1(t) \leq Ct^{-1}. \tag{4.3}$$

Proof. Here, we add the second-order energy to the modified energy functional V_1; that is,

$$V_2 := V_1 + ME_2.$$

We have

$$\begin{aligned} V_2'(t) &= V_1'(t) + ME_2'(t) \\ &\leq V_1'(t) + \mathcal{D}M_5 \left[\varepsilon_{10} \|\psi_x\|_2^2 + \frac{1}{2\kappa^2 \varepsilon_{10}} \left(\tau_0^2 \|\sigma_t\|_2^2 + \delta^2 \|\sigma\|_2^2 \right) \right] - M\frac{\delta}{\kappa} \|\sigma_t\|_2^2 . \end{aligned} \tag{4.4}$$

According to the discussion above on V_1' to choose the different parameters and the fact that ε_{10} may be selected as small as we wish, it appears that, for a larger value of M, (4.4) implies that

$$V_2' \leq -C_1 E_1,$$

for some positive constant C_1. An integration with respect to time of this inequality, taking into account that E_1 is non-increasing yields

$$tE_1(t) \leq \int_0^t E_1(s)ds \leq \frac{V_2(0)}{C_1}.$$

This completes the proof of (4.3).

5. General remarks

Remark 1. We suspect that the condition $\mathcal{D} = 0$ is necessary for exponential stability (3.30), but this should be proved. As an adequate method to prove this kind of results, the spectral theory which gives a direct connection between the exponential stability and the spectrum of operators generated by hyperbolic systems; see for example [17]. This method is completely different from the one used in the present paper.

Remark 2. The mass can be deleted; that is $m_E = 0$. In this case, the last line in $\mathcal{B}\Psi$, the variable y and its space $\mathbb{R}$ are not considered, and the relation $y = w(L, \cdot)$ (in the definition of $D(\mathcal{B})$) is replaced by

$$K(\chi_x(L,t) + \psi(L,t)) + \mu w(L,t) = 0.$$

Our well-posedness and stability results hold true.

Remark 3. Our well-posedness and stability results are valid for the following class of feedback control forces:

$$\tau(t) = -K\chi_t(0,t) - \tilde{\mu}\chi_t(L,t) - \chi(0,t), \tag{5.1}$$

where $\tilde{\mu} \in \mathbb{R}$ satisfying

$$(\mu - \tilde{\mu})^2 \leq 4\mu K. \tag{5.2}$$

The feedback control is applied at the base. The terms $\chi(0,t)$ and $\chi(L,t)$ can be measured by displacement sensing devices whilst $\chi_t(0,t)$ and $\chi_t(L,t)$ are computed by a backward difference algorithm of the values of $\chi(0,t)$ and $\chi(L,t)$, respectively. In case (5.1), we have instead of (2.22)

$$E_1'(t) = -K\chi_t^2(0,t) - \mu\chi_t^2(L,t) + (\mu - \tilde{\mu})\chi_t(L,t)\chi_t(0,t) - \frac{\delta}{\kappa}\|\sigma\|_2^2,$$

so, for any real number ϵ satisfying

$$\frac{(\mu - \tilde{\mu})^2}{2\mu} < |\mu - \tilde{\mu}|\epsilon < 2K \tag{5.3}$$

(ϵ exists according to (5.2)), we get

$$E_1'(t) \leq -\left(K - \frac{|\mu - \tilde{\mu}|\epsilon}{2}\right)\chi_t^2(0,t) - \left(\mu - \frac{|\mu - \tilde{\mu}|}{2\epsilon}\right)\chi_t^2(L,t) - \frac{\delta}{\kappa}\|\sigma\|_2^2 .$$

Because $K - \frac{|\mu - \tilde{\mu}|\epsilon}{2} > 0$ and $\mu - \frac{|\mu - \tilde{\mu}|}{2\epsilon} > 0$ (thanks to (5.3)), the proofs of the well-posedness (with small modification of the operator $\mathcal{B}$) and stability results are very similar.

Mathematically, (5.2) means that

$$\mu - 2\sqrt{\mu K} \leq \tilde{\mu} \leq \mu + 2\sqrt{\mu K}.$$

This means, roughly, that if the control gain K is important, then any 'small' coefficient $\tilde{\mu}$ should be enough. Whereas, if the control gain K is small, there is a threshold for $\tilde{\mu}$.

Remark 4. Our well-posedness and stability results are still satisfied if we add the term $\lambda\theta$ to the fourth equation in (2.8), where λ is a positive constant. In this case, the derivative of E_1 satisfies

$$E_1'(t) = -K\chi_t^2(0,t) - \mu\chi_t^2(L,t) - \frac{\delta}{\kappa}\|\sigma\|_2^2 - \lambda\|\theta\|_2^2. \tag{5.4}$$

We see that the term $\lambda\theta$ generates the last dissipation in (5.4), and then the situation becomes more favorable mathematically. We can take either $K = 0$ or $\mu = 0$.

Remark 5. The results of this paper can be extended easily to the Fourier's law case; that is $\tau_0 = 0$. Indeed, from the last equation in (2.8), σ can be explicitly expressed in term of θ by $\sigma = -\frac{\kappa}{\delta}\theta_x$. Then, the fourth equation in (2.8) can be transformed into the following heat equation in term of θ:

$$\rho_3\theta_t - \frac{\kappa}{\delta}\theta_{xx} + \gamma\chi_{tx} = 0. \tag{5.5}$$

In this case, the last two equations in (2.8) will be replaced by (5.5), σ will not be considered in the definition of Ψ, and thanks to the homogeneous Dirichlet boundary conditions on θ in (2.3), the derivative of E_1 becomes

$$E_1'(t) = -K\chi_t^2(0,t) - \mu\chi_t^2(L,t) - \frac{\kappa}{\delta}\|\theta_x\|_2^2. \tag{5.6}$$

According to Poincaré's inequality, $\|\theta\|_2^2$ is dominated by $\|\theta_x\|_2^2$, and so the dissipation (5.6) is strong enough to stabilize the system (and even we can take $K = 0$ or $\mu = 0$).

Remark 6. The stability estimate (4.3) is satisfied only for classical solutions ($\Psi_0 \in D(\mathcal{B})$) and it can not be extended to weak solutions ($\Psi_0 \in \mathcal{H}$) by density arguments, since the constant C in (4.3) depends on E_2, which is not defined for weak solutions even in the distributions sense.

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