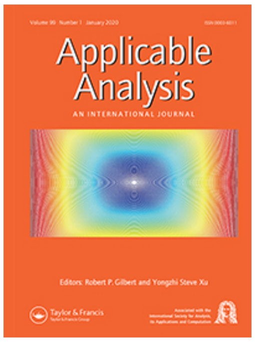

Applicable Analysis
An International Journal

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: https://www.tandfonline.com/loi/gapa20

Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity

Mohammad M. Al-Gharabli, Aissa Guesmia & Salim A. Messaoudi

To cite this article: Mohammad M. Al-Gharabli, Aissa Guesmia & Salim A. Messaoudi (2020) Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity, Applicable Analysis, 99:1, 50-74, DOI: 10.1080/00036811.2018.1484910

To link to this article: https://doi.org/10.1080/00036811.2018.1484910

APPLICABLE ANALYSIS
2020, VOL. 99, NO. 1, 50–74
https://doi.org/10.1080/00036811.2018.1484910

Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity

Mohammad M. Al-Gharabli[a], Aissa Guesmia[b] and Salim A. Messaoudi[c]

[a]The Preparatory Year Program, Department of Mathematics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia; [b]Institut Elie Cartan de Lorraine, UMR 7502 Université de Lorraine, Metz, France; [c]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

ABSTRACT
In this paper, we consider a viscoelastic plate equation with a velocity-dependent material density and a logarithmic nonlinearity. Using the Faedo-Galaerkin approximations and the multiplier method, we establish the existence of the solutions of the problem and we prove an explicit and general decay rate result. These results extend and improve many results in the literature.

ARTICLE HISTORY
Received 10 January 2018
Accepted 1 June 2018

COMMUNICATED BY
Rolando Magnanini

KEYWORDS
Asymptotic stability; viscoelastic; plate equation; logarithmic nonlinearity

AMS SUBJECT CLASSIFICATIONS
35B35 ; 35L55; 75D05; 74D10; 93D20

1. Introduction

In this paper, we deal with the existence and decay of solutions of the following plate problem:

$$
\begin{aligned}
&|u_t|^{\rho} u_{tt} + \Delta^2 u + \Delta^2 u_{tt} - \int_0^t g(t-s)\Delta^2 u(s)\,\mathrm{d}s = ku\ln|u|, \quad \text{in } \Omega\times(0,\infty),\\
&u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad \text{in } \partial\Omega\times(0,\infty),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{aligned} \tag{1}
$$

where Ω is a bounded domain of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$ and ρ and k are positive constants. The kernel g is satisfying some conditions to be specified later.

1.1. *Problems with a velocity-dependent material density*

Cavalcanti et al. [1] considered

$$
\begin{aligned}
&|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s - \gamma\Delta u_t = 0, \quad \text{in } \Omega\times(0,\infty),\\
&u(x,t) = 0, \quad \text{in } \partial\Omega\times(0,\infty),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{aligned} \tag{2}
$$

CONTACT Mohammad M. Al-Gharabli mahfouz@kfupm.edu.sa

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, ρ is a positive real number satisfying some conditions and g is a positive exponentially decaying function. They established a global existence result when the constant $\gamma \geq 0$, and an exponential decay result for the case $\gamma > 0$. Messaoudi and Tatar [2] extended this decay result to the case where a source term is competing with the viscoelastic and the strong damping. In the absence of the strong damping ($\gamma = 0$), Messaoudi and Tatar [3,4] studied (2) and showed that the viscoelastic damping is strong enough to drive the system uniformly to rest. Precisely, they showed that the energy of the solution decays exponentially (resp. polynomially) if g decays exponentially (resp. polynomially). Later, Han and Wang [5] considered (2) for $\gamma = 0$ and with a relaxation function of more general decay type and established, similarly to the work of Messaoudi [6,7], a general decay result in which the usual exponential and the polynomial decay are only special cases. Liu [8] considered (2), for $\gamma = 0$, and in the presence of a source term. He established a general decay result similar to the one in [5]. In [9], Liu studied the problem

$$\begin{aligned}
&|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + \alpha(t)h(u_t) = b|u|^{p-2}u, \quad \text{in } \Omega \times (0,\infty),\\
&u(x,t) = 0, \quad \text{in } \partial\Omega \times (0,\infty),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{aligned} \tag{3}$$

and proved, without imposing growth conditions on h, a general decay result which depends on the behavior of g, α and h. Messaoudi and Mustafa [10] studied (2) for relaxation functions satisfying

$$g'(t) \leq -H(g(t)), \tag{4}$$

where $H \in \mathbb{C}^1(\mathbb{R}^+)$, with $H(0) = 0$ and H is linear or strictly increasing and strictly convex function $\mathbb{C}^2$ near the origin. They obtained an explicit and general relation between the decay rate for the energy and that of the relaxation function g without imposing restrictive assumptions on the behavior of g at infinity. Recently, Cavalcanti et al. [11] considered (2), with $\gamma = 0$, and a relaxation function satisfying (4). In addition, they required

$$\liminf_{x\to 0^+} \{x^2H''(x) - xH'(x) + H(x)\} \geq 0$$

and some other condition and proved that the energy uniformly decays to zero with the rate that is determined from the solutions of the ODE quantifying the behavior of $g(t)$.

Very recently, Messaoudi and Al-Khulaifi [12] considered (2), with $\gamma = 0$, where the relaxation function satisfies (11) below and established an optimal and general decay result.

1.2. Plate Problems

Concerning the study of plates, Lagnese [13] studied a viscoelastic plate equation and showed that the energy decays to zero as time goes to infinity by intorducing a dissipative mechanism on the boundary of the system. Rivera et al. [14] proved that the first- and second-order energy, associated with the solutions of the viscoelastic plate equation, decay exponentially provided that the kernel of the memory also decays exponentially. Komornik [15] investigated the energy decay of a plate model under weak growth assumptions on the feedback function. Messaoudi [16] studied the following problem:

$$\begin{aligned}
&u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u, \quad \text{in } Q_T = \Omega \times (0,T),\\
&u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \Gamma_T = \partial\Omega \times (0,T),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{aligned} \tag{5}$$

and established an existence result and showed that the solution continues to exist globally if $m \geq p$, and blows up in finite time if $m < p$ and the initial energy is negative. This result was later improved by Chen and Zhou [17].

For boundary damping, Santos and Junior [18] studied the stability of the following problem:

$$\begin{aligned}
&u_{tt} + \Delta^2 u = 0, \quad \text{in } \Omega \times (0,\infty),\\
&u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \Gamma_0 \times (0,\infty),\\
&-u + \int_0^t g_1(t-s)\beta_1 u(s)\,ds = 0, \quad \text{on } \Gamma_1 \times (0,\infty),\\
&\frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\beta_2 u(s)\,ds = 0, \quad \text{on } \Gamma_2 \times (0,\infty),\\
&u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \text{ in } \Omega,
\end{aligned} \tag{6}$$

where

$$\beta_1 u = \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \beta_2 u = \frac{\partial \Delta u}{\partial \mu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

with

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} \quad \text{and} \quad B_2 u = (\nu_1 - \nu_2)u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx}).$$

For more results in this direction, see [19–23].

1.3. Problems with logarithmic nonlinearity

The logarithmic nonlinearity is of much interest in physics, since it appears naturally in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics [24,25]. This type of problems has applications in many branches of physics such as nuclear physics, optics and geophysics [26–28]. Birula and Mycielski [27,29] studied the following problem:

$$\begin{aligned}
&u_{tt} - u_{xx} + u - \varepsilon u \ln|u|^2 = 0, \quad \text{in } [a,b] \times (0,T),\\
&u(a,t) = u(b,t) = 0, \quad \text{in } (0,T),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \text{ in } [a,b],
\end{aligned} \tag{7}$$

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit $p \to 1$ for the p-adic string equation [30,31]. In [32], Cazenave and Haraux considered

$$u_{tt} - \Delta u = u \ln|u|^k, \quad \text{in } \mathbb{R}^3 \tag{8}$$

and established the existence and uniqueness of the solution for the Cauchy problem. Gorka [28] used some compactness arguments and obtained the global existence of weak solutions, for all

$$(u_0, u_1) \in H_0^1([a,b]) \times L^2([a,b]),$$

to the initial-boundary value problem (8) in the one-dimensional case. Bartkowski and Gorka [26] proved the existence of classical solutions and investigated the weak solutions for the corresponding one-dimensional Cauchy problem for Equation (8). Hiramatsu et al. [33] introduced the following equation:

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u| \tag{9}$$

to study the dynamics of Q-ball in theoretical physics and presented a numerical study. However, there was no theoretical analysis for the problem. In [34], Han proved the global existence of weak solutions, for all

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega),$$

to the initial boundary value problem (9) in $\mathbb{R}^3$.

In this paper, we are concerned with the well-posedness and stability of the plate problem (1) with kernels g having an arbitrary growth at infinity (condition (11) below). The obtained stability results improve and generalize many results in the literature.

This paper is organized as follows. In Section 2, we present some notations and material needed for our work. In Section 3, we establish the local existence of the solutions of the problem. The global existence and the decay results are presented in Sections 4 and 5, respectively.

2. Preliminaries

In this section, we present some material needed for the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant.

We consider the following hypotheses:

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1- nonincreasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\,\mathrm{d}s = \ell > 0. \tag{10}$$

(A2) There exist a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\xi(0) > 0$, and a constant $1 \le p < \frac{3}{2}$ such that

$$g'(t) \le -\xi(t) g^p(t), \quad \forall t \in \mathbb{R}^+. \tag{11}$$

(A3) The constant k in (1) is such that

$$0 < k < k_0 = \frac{2\pi \ell e^3}{c_p}, \tag{12}$$

where c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \le c_p \|\Delta u\|_2^2, \quad \forall u \in H_0^2(\Omega),$$

where $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)}$.

The energy functional associated with problem (1) is

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(\left(1-\int_0^t g(s)\,\mathrm{d}s\right)\|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 - k\int_\Omega u^2 \ln|u|\,\mathrm{d}x\right)$$
$$+\frac{k}{4}\|u\|_2^2 + \frac{1}{2}(go\Delta u), \tag{13}$$

where

$$(go\Delta u)(t) = \int_0^t g(t-s)\|\Delta u(s) - \Delta u(t)\|_2^2\,\mathrm{d}s.$$

Direct differentiation of (13), using (1), leads to

$$E'(t) = \frac{1}{2}(g'o\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \le \frac{1}{2}(g'o\Delta u)(t) \le 0. \tag{14}$$

Lemma 2.1 ([35,36]): (*Logarithmic Sobolev inequality*) *Let u be any function in $H_0^1(\Omega)$ and $a > 0$ be any number. Then*

$$\int_\Omega u^2 \ln|u|\,\mathrm{d}x \le \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{a^2}{2\pi}\|\nabla u\|_2^2 - (1+\ln a)\|u\|_2^2. \tag{15}$$

Corollary 2.2: *Let u be any function in $H_0^2(\Omega)$ and $a > 0$ be any number. Then*

$$\int_\Omega u^2 \ln|u|\,\mathrm{d}x \le \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{c_p a^2}{2\pi}\|\Delta u\|_2^2 - (1+\ln a)\|u\|_2^2. \tag{16}$$

Lemma 2.3 ([32]): (*Logarithmic Gronwall inequality*) *Let $C > 0$, $\gamma \in L^1(0,T;\mathbb{R}^+)$ and assume that the function $w : [0,T] \to [1,\infty)$ satisfies*

$$w(t) \le C\left(1 + \int_0^t \gamma(s)w(s)\ln(w(s))\,\mathrm{d}s\right), \quad \forall t \in [0,T]. \tag{17}$$

Then

$$w(t) \le C\exp\left(C\int_0^t \gamma(s)\,\mathrm{d}s\right), \quad \forall t \in [0,T]. \tag{18}$$

Lemma 2.4: *Let $\epsilon_0 \in (0,1)$. Then there exists $d_{\varepsilon_0} > 0$ such that*

$$s|\ln s| \le s^2 + d_{\epsilon_0} s^{1-\epsilon_0}, \quad \forall s > 0. \tag{19}$$

Proof: Let $r(s) = s^{\epsilon_0}(|\ln s| - s)$. Notice that r is continuous on $(0,\infty)$ and its limit at 0^+ is 0^+, and its limit at ∞ is $-\infty$. Then r has a maximum d_{ϵ_0} on $(0,\infty)$, so (19) holds. ∎

3. Local existence

In this section, we state and prove the local existence result for problem (1).

Definition 3.1: Let $T > 0$. A function

$$u \in C^1([0,T], H_0^2(\Omega))$$

is called a weak solution of (1) on $[0, T]$ if

$$\begin{aligned} &\int_\Omega |u_t|^\rho u_{tt}(x,t)w(x)\,\mathrm{d}x + \int_\Omega \Delta u(x,t)\Delta w(x)\,\mathrm{d}x + \int_\Omega \Delta u_{tt}\Delta w\,\mathrm{d}x \\ &\quad - \int_\Omega \Delta w(x)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\,\mathrm{d}x = k\int_\Omega w(x)u(x,t)\ln|u(x,t)|\,\mathrm{d}x, \quad \forall w \in H_0^2(\Omega), \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{aligned} \tag{20}$$

Theorem 3.2: *Assume that* (A1)–(A3) *hold and let* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$. *Then problem (1) has a weak solution on* $[0, T]$.

Proof: To establish the existence of a solution to problem (1), we use the Faedo-Galerkin approximations. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the 'separable' space $H_0^2(\Omega)$. Let $V_m = \mathrm{span}\{w_1, w_2, \ldots, w_m\}$ and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$u_0^m(x) = \sum_{j=1}^m a_j w_j(x), \quad u_1^m(x) = \sum_{j=1}^m b_j w_j(x),$$

where

$$u_0^m \to u_0 \text{ in } H_0^2(\Omega) \quad \text{and} \quad u_1^m \to u \text{ in } H_0^2(\Omega), \text{ as } m \to \infty. \tag{21}$$

We search for an approximate solution

$$u^m(x,t) = \sum_{j=1}^m h_j^m(t) w_j(x)$$

of the approximate problem in V_m

$$\begin{aligned} &\int_\Omega \left(|u_t^m|^\rho u_{tt}^m w + \Delta u^m \Delta w + \Delta u_{tt}^m \Delta w - \int_0^t g(t-s)\Delta u^m(s)\Delta w ds \right) \mathrm{d}x \\ &\quad = k\int_\Omega w u^m \ln|u^m|\,\mathrm{d}x, \quad \forall w \in V_m, \\ &u^m(0) = u_0^m = \sum_{j=1}^m (u_0, w_j) w_j, \\ &u_t^m(0) = u_1^m = \sum_{j=1}^m (u_1, w_j) w_j. \end{aligned} \tag{22}$$

This leads to a system of ODEs for unknown functions $h_j^m(t)$. Based on standard existence theory for ODE, one can obtain functions

$$h_j : [0, t_m) \to \mathbb{R}, \quad j = 1, 2, \ldots, m,$$

which satisfy (22) in a maximal interval $[0, t_m)$, $t_m \in (0, T]$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independently of m and t. For this purpose, let $w = u_t^m$ in (22) and integrate by parts to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E^m(t) \le \frac{1}{2}(g' o \Delta u^m) \le 0, \tag{23}$$

where

$$\begin{aligned} E^m(t) &= \frac{1}{\rho+2}\|u_t^m\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(\|\Delta u_t^m\|_2^2 + \left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\Delta u^m\|_2^2 + (go\Delta u^m)(t)\right) \\ &\quad + \frac{k}{4}\|u^m\|_2^2 - \frac{k}{2}\int_\Omega |u^m|^2 \ln|u^m|\,\mathrm{d}x. \end{aligned} \tag{24}$$

From (23), we have

$$E^m(t) \le E^m(0), \quad \forall t \ge 0.$$

The last inequality together with the Logarithmic Sobolev inequality leads to

$$\begin{aligned} &\|u_t^m\|_{\rho+2}^{\rho+2} + \|\Delta u_t^m\|_2^2 + \left(\ell - \frac{ka^2c_p}{2\pi}\right)\|\Delta u^m\|_2^2 + \left[\frac{k}{2} + k(1+\ln a)\right]\|u^m\|_2^2 + go\Delta u^m \\ &\le C + \|u^m\|_2^2 \ln\|u^m\|_2^2, \end{aligned} \tag{25}$$

where $C = 2E^m(0)$. Choosing

$$\mathrm{e}^{-3/2} < a < \sqrt{\frac{2\pi\ell}{kc_p}} \tag{26}$$

will make

$$\ell - \frac{ka^2c_p}{2\pi} > 0$$

and

$$\frac{k}{2} + k(1+\ln a) > 0.$$

This selection is possible thanks to (A3). So, we get

$$\|u_t^m\|_{\rho+2}^{\rho+2} + \|\Delta u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 + go\Delta u^m \le c(1 + \|u^m\|_2^2 \ln\|u^m\|_2^2). \tag{27}$$

Let us note that

$$u^m(\cdot,t) = u^m(\cdot,0) + \int_0^t \frac{\partial u^m}{\partial s}(\cdot,s)\,\mathrm{d}s.$$

Then, using Cauchy-Schwarz' inequality, we get

$$\begin{aligned} \|u^m(t)\|_2^2 &\le 2\|u^m(0)\|_2^2 + 2\left\|\int_0^t \frac{\partial u^m}{\partial s}(s)\,\mathrm{d}s\right\|_2^2 \\ &\le 2\|u^m(0)\|_2^2 + 2T\int_0^t \|u_t^m(s)\|_2^2\,\mathrm{d}s, \end{aligned} \tag{28}$$

hence, inequality (27) gives

$$\|u^m\|_2^2 \le 2\|u^m(0)\|_2^2 + 2Tc\left(1 + \int_0^t \|u^m\|_2^2 \ln\|u^m\|_2^2\,\mathrm{d}s\right). \tag{29}$$

If we put $C_1 = \max\{2Tc, 2\|u(0)\|_2^2\}$, (29) leads to

$$\|u^m\|_2^2 \le 2C_1\left(1 + \int_0^t \|u^m\|_2^2 \ln(\|u^m\|_2^2)\,\mathrm{d}s\right).$$

Because $C_1 \ge 0$, we get

$$\|u^m\|_2^2 \le 2C_1\left(1 + \int_0^t (C_1 + \|u^m\|_2^2)\ln(C_1 + \|u^m\|_2^2)\,\mathrm{d}s\right).$$

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate:

$$\|u^m\|_2^2 \le 2C_1 e^{2C_1 T} = C_2.$$

Hence, from inequality (27) it follows that

$$(go\Delta u^m)(t) + \|u_t^m\|_{\rho+2}^{\rho+2} + \|\Delta u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \le c(1 + C_2 \ln C_2) = C_3.$$

This implies

$$\sup_{t\in(0,t_m)} [(go\Delta u^m)(t) + \|u_t^m\|_{\rho+2}^{\rho+2} + \|\Delta u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2] \le C_3. \tag{30}$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T.

Substituting $w = u_{tt}^m$ in (22) and using Young's and Cauchy-Schwarz' inequalities, we obtain

$$\begin{aligned}\int_\Omega |u_t^m|^\rho |u_{tt}^m|^2\,\mathrm{d}x + \|\Delta u_{tt}^m\|_2^2 &= -\int_\Omega \Delta u^m \Delta u_{tt}^m\,\mathrm{d}x + \int_\Omega\int_0^t g(t-s)\Delta u^m(s)\Delta u_{tt}^m(t)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad + k\int_\Omega u_{tt}^m u^m \ln|u^m|\,\mathrm{d}x \\ &\le \delta\|\Delta u_{tt}^m\|_2^2 + \frac{1}{4\delta}\left(\int_0^t g(t-s)\|\Delta u^m(s)\|_2\,\mathrm{d}s\right)^2 + \delta\|\Delta u_{tt}^m\| \\ &\quad + \frac{1}{4\delta}\|\Delta u^m(t)\|^2 + k\int_\Omega u_{tt}^m u^m \ln|u^m|\,\mathrm{d}x. \end{aligned} \tag{31}$$

To estimate the last term in the right-hand side of (31), we apply (19) with $\varepsilon_0 = \frac{1}{2}$ and use repeatedly Young's, Cauchy-Schwarz' and the embedding inequalities as follows

$$\begin{aligned} k\int_\Omega u_{tt}^m u^m \ln|u^m|\,\mathrm{d}x &\le c\int_\Omega u_{tt}^m\left(|u^m|^2 + d_2\sqrt{u^m}\right)\mathrm{d}x \\ &\le c\left(\delta\int_\Omega |u_{tt}^m|^2\,\mathrm{d}x + \frac{1}{4\delta}\int_\Omega (|u^m|^2 + d_2\sqrt{u^m})^2\,\mathrm{d}x\right) \\ &\le c\delta\|\Delta u_{tt}^m\|_2^2 + \frac{c}{4\delta}\left(\int_\Omega |u^m|^4\,\mathrm{d}x + \int_\Omega |u^m|\,\mathrm{d}x\right) \\ &\le c\delta\|\Delta u_{tt}^m\|_2^2 + \frac{c}{4\delta}(\|\Delta u^m\|_2^4 + \|u^m\|_2). \end{aligned} \tag{32}$$

Combining (31) and (32) to have

$$\begin{aligned}\int_\Omega |u_t^m|^\rho |u_{tt}^m|^2\,\mathrm{d}x + (1 - c\delta)\|\Delta u_{tt}^m\|_2^2 &\le \frac{1}{4\delta}\left(\int_0^t g(t-s)\|\Delta u^m(s)\|_2\,\mathrm{d}s\right)^2 \\ &\quad + \frac{1}{4\delta}\|\Delta u^m\|^2 + \frac{c}{\delta}(\|\Delta u^m\|_2^4 + \|u^m\|_2^2). \end{aligned} \tag{33}$$

Integrate the last inequality on $(0, T)$ and use (10) and (30), we obtain

$$\int_0^T \int_\Omega |u_t^m|^\rho |u_{tt}^m|^2 \,dx\,dt + (1 - c\delta) \int_0^T \|\Delta u_{tt}^m\|_2^2 \,dt$$
$$\leq \frac{c}{\delta} \int_0^T [(go\Delta u^m)(t) + \|\Delta u^m\|_2^2 + \|\Delta u^m\|_2^4 + \|u^m\|_2^2] \,dt. \tag{34}$$

From the last inequality, choosing $\delta > 0$ small enough and using (30), we get the following, for some positive constant C_4 not depending neither on m nor on t:

$$\int_0^T \|\Delta u_{tt}^m\|_2^2 \,dt \leq C_4. \tag{35}$$

From (30) and (35), we have

$$\begin{aligned} &(u^m) \text{ is uniformly bounded in } L^\infty(0, T; H_0^2(\Omega)),\\ &(u_t^m) \text{ is uniformly bounded in } L^\infty(0, T; L^{\rho+2}(\Omega)) \cap L^\infty(0, T; H_0^2(\Omega)),\\ &(u_{tt}^m) \text{ is uniformly bounded in } L^2(0, T; H_0^2(\Omega)), \end{aligned} \tag{36}$$

which implies that there exists a subsequence of (u^m) (still denoted by (u^m)), such that

$$\begin{aligned} &u^m \rightharpoonup u \text{ weakly} * \text{ in } L^\infty(0, T; H_0^2(\Omega)),\\ &u_t^m \rightharpoonup u_t \text{ weakly} * \text{ in } L^\infty(0, T; L^{\rho+2}(\Omega)) \cap L^\infty(0, T; H_0^2(\Omega)),\\ &u^m \rightharpoonup u \text{ weakly in } L^2(0, T; H_0^2(\Omega)),\\ &u_t^m \rightharpoonup u_t \text{ weakly in } L^2(0, T; L^{\rho+2}(\Omega)) \cap L^2(0, T; H_0^2(\Omega)),\\ &u_{tt}^m \rightharpoonup^w u_{tt} \text{ in } L^2(0, T; H_0^2(\Omega)). \end{aligned} \tag{37}$$

Analysis of the non-linear terms

(1) *Term $u^m \ln |u^m|$*: using (36), we have (u^m) is bounded in $L^\infty(0, T; H_0^2(\Omega))$ which implies, using the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ ($\Omega \subset \mathbb{R}^2$), the boundedness of (u^m) in $L^2(\Omega \times (0, T))$. Similarly, (u_t^m) is bounded in $L^2(\Omega \times (0, T))$. Then, making use of Aubin-Lions' theorem, we find, up to a subsequence, that

$$u^m \to u \quad \text{strongly in } L^2(\Omega \times (0, T))$$

and

$$u^m \to u \quad \text{a.e. in } \Omega \times (0, T).$$

Since the maps $s \to ks \ln |s|$ is continuous, we have the following convergence:

$$ku^m \ln |u^m| \to ku \ln |u| \quad \text{a.e. in } \Omega \times (0, T). \tag{38}$$

Using the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ ($\Omega \subset \mathbb{R}^2$), it is clear that $k(u^m \ln |u^m|)$ is bounded in $L^\infty(\Omega \times (0, T))$. Next, taking into account the Lebesgue bounded convergence theorem (Ω is bounded), we get

$$ku^m \ln |u^m| \to ku \ln |u| \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \tag{39}$$

(2) *Term* $|u_t^m|^\rho u_t^m$: using (35), we have (u_t^m) is uniformly bounded in $L^\infty(0,T;H_0^2(\Omega))$ which implies the boundedness of (u_t^m) in $L^\infty(\Omega\times(0,T))$, and so in $L^2(\Omega\times(0,T))$. Using (35), we see that (u_{tt}^m) is bounded in $L^2((0,T);H_0^2(\Omega))$ which implies that (u_{tt}^m) is bounded in $L^2(\Omega\times(0,T))$. Now, using Aubin-Lions theorem, there exists a subsequence, still denoted by (u_t^m), such that

$$u_t^m \to u_t \quad \text{strongly in } L^2(0,T;L^2(\Omega))$$

and

$$|u_t^m|^\rho u_t^m \to |u_t|^\rho u_t \quad \text{a.e. in } \Omega\times(0,T). \tag{40}$$

Using (30) and the embedding theorems, we have

$$\begin{aligned}\||u_t^m|^\rho u_t\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \|u_t^m\|_{2(\rho+1)}^{2(\rho+1)}\,\mathrm{d}t\\ &\le c\int_0^T \|\Delta u_t^m(t)\|_2^{2(\rho+1)}\,\mathrm{d}t \le cTC_3^{\rho+1},\end{aligned} \tag{41}$$

which implies that $(|u_t^m|^\rho u_t^m)$ is bounded in $L^2(\Omega\times(0,T))$. Combining (40) and (41) and using Lions' lemma, see Lions ([37], pp. 12), we obtain

$$|u_t^m|^\rho u_t^m \rightharpoonup |u_t|^\rho u_t \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{42}$$

Now, we integrate (22) over $(0,t)$ to obtain, for every $w\in V_m$,

$$\begin{aligned}&\frac{1}{\rho+1}\int_\Omega |u_t^m|^\rho u_t^m w\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\rho+1}\int_\Omega |u_1^m|u_1^m w\,\mathrm{d}x + \int_0^t\int_\Omega \Delta u^m(s)\Delta w\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_\Omega \Delta u_t^m\Delta w\,\mathrm{d}x\mathrm{d}s - \int_\Omega \Delta u_1^m \Delta w\,\mathrm{d}x - \int_\Omega\int_0^t\left(\int_0^\tau g(\tau-s)\Delta u^m(s)\right)\Delta w\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}x\\ &= k\int_0^t\int_\Omega wu^m(s)\ln|u^m(s)|\,\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{43}$$

Convergences (21), (37), (39) and (42) are sufficient to pass to the limit in (43) as $m\to\infty$, and get, for any $w\in V_m$ and $m\ge 1$,

$$\begin{aligned}\frac{1}{\rho+1}\int_0^t\int_\Omega |u_s|^\rho u_s w\,\mathrm{d}x\mathrm{d}s &= \frac{1}{\rho+1}\int_\Omega |u_1^m|u_1^m w\,\mathrm{d}x - \int_0^t\int_\Omega \Delta u(s)\Delta w\,\mathrm{d}x\mathrm{d}s\\ &\quad - \int_\Omega \Delta u_t\Delta w\,\mathrm{d}x\mathrm{d}s + \int_\Omega \Delta u_1^m\Delta w\,\mathrm{d}x\\ &\quad + \int_\Omega\int_0^t\left(\int_0^\tau g(\tau-s)\Delta u(s)\right)\Delta w\,\mathrm{d}s\,\mathrm{d}\tau\,\mathrm{d}x\\ &\quad + k\int_0^t\int_\Omega wu(s)\ln|u(s)|\,\mathrm{d}x\mathrm{d}s,\end{aligned} \tag{44}$$

which implies that (44) is valid for any $w\in H_0^2(\Omega)$. Using the fact that the terms in the right-hand side of (44) are absolutely continuous (since they are functions of t defined by integrals over $(0,t)$), then (44) is differentiable for a.e. $t\in\mathbb{R}^+$. Thus, differentiating (44), we obtain, for a.e. $t\in(0,T)$

and $w \in H_0^2(\Omega)$,

$$\begin{aligned}
&\int_\Omega |u_t|^\rho u_{tt} w \,\mathrm{d}x\mathrm{d}s + \int_\Omega \Delta u(t)\Delta w \,\mathrm{d}x \\
&\quad + \int_\Omega \Delta u_{tt} \Delta w \,\mathrm{d}x - \int_\Omega \left(\int_0^t g(t-s)\Delta u(s)\right) \Delta w \,\mathrm{d}s\,\mathrm{d}x \\
&= k \int_\Omega w u(t) \ln |u(t)| \,\mathrm{d}x\mathrm{d}s. \qquad (45)
\end{aligned}$$

This ends the proof of Theorem 3.2. ∎

4. Global Existence

In this section, we state and prove a global existence result under smallness conditions on the initial data (u_0, u_1). For this purpose, we introduce the following functionals:

$$\begin{aligned}
J(t) = \frac{1}{2}&\left(\left(1 - \int_0^t g(s)\,\mathrm{d}s\right) \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + go\Delta u - k \int_\Omega u^2 \ln |u| \,\mathrm{d}x\right) \\
&+ \frac{k}{4}\|u\|_2^2 \qquad (46)
\end{aligned}$$

and

$$I(t) = \left(1 - \int_0^t g(s)\,\mathrm{d}s\right) \|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + go\Delta u - 3k \int_\Omega u^2 \ln |u| \,\mathrm{d}x. \qquad (47)$$

Lemma 4.1: *The following inequalities hold*:

$$-kd_0\sqrt{|\Omega| c_*^3} \|\Delta u\|_2^{3/2} \le k \int_\Omega u^2 \ln |u| \,\mathrm{d}x \le kc_*^3 \|\Delta u\|_2^3, \quad \forall u \in H_0^2(\Omega), \qquad (48)$$

where $d_0 = \sup_{0<s<1} \sqrt{s} |\ln s| = \frac{2}{e}$, $|\Omega|$ *is the Lesbegue measure of* Ω *and* c_* *is the smallest embedding constant*

$$\left(\int_\Omega |u|^3 \,\mathrm{d}x\right)^{1/3} \le c_* \|\Delta u\|_2, \quad \forall u \in H_0^2(\Omega) \qquad (49)$$

(c_* *exists thanks to the embedding of* $H_0^2(\Omega)$ *in* $L^\infty(\Omega)$ *and* $\Omega \subset \mathbb{R}^2$*).*

Proof: Let

$$\Omega_1 = \{x \in \Omega : |u(x)| > 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |u(x)| \le 1\}.$$

So, using (49), we have

$$\begin{aligned}
k \int_\Omega u^2 \ln |u| \,\mathrm{d}x &= k \int_{\Omega_2} u^2 \ln |u| \,\mathrm{d}x + k \int_{\Omega_1} u^2 \ln |u| \,\mathrm{d}x \\
&\le k \int_{\Omega_1} u^2 \ln |u| \,\mathrm{d}x \le k \int_{\Omega_1} |u|^3 \,\mathrm{d}x \le k \int_\Omega |u|^3 \,\mathrm{d}x \le kc_*^3 \|\Delta u\|_2^3.
\end{aligned}$$

On the other hand, using Hölder's inequality and (49), we find

$$\begin{aligned}-k\int_{\Omega} u^2 \ln|u|\,\mathrm{d}x &= -k\int_{\Omega_2} u^2 \ln|u|\,\mathrm{d}x - k\int_{\Omega_1} u^2 \ln|u|\,\mathrm{d}x\\ &\le -k\int_{\Omega_2} u^2 \ln|u|\,\mathrm{d}x = k\int_{\Omega_2} u^2|\ln|u||\,\mathrm{d}x\\ &\le kd_0\int_{\Omega}|u|^{3/2}\,\mathrm{d}x \le kd_0\sqrt{|\Omega|}\left(\int_{\Omega}|u|^3\,\mathrm{d}x\right)^{1/2} \le kd_0\sqrt{|\Omega|c_*^3}\|\Delta u\|_2^{3/2},\end{aligned}$$

which implies the left inequality in (48). ■

Lemma 4.2: *Assume that* (A1)–(A3). *Let* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *such that*

$$I(0) > 0 \quad \text{and} \quad \sqrt{54}kc_*^3\left(\frac{E(0)}{\ell}\right)^{1/2} < \ell. \tag{50}$$

Then

$$I(t) > 0, \quad \forall t \in [0, T). \tag{51}$$

Proof: From (47), we have

$$k\int_{\Omega} u^2 \ln|u|\,\mathrm{d}x = \frac{1}{3}\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\Delta u\|_2^2 + \frac{1}{3}\|\Delta u_t\|_2^2 + \frac{1}{3}go\Delta u - \frac{1}{3}I(t). \tag{52}$$

Substitute (52) in (46), we find

$$J(t) = \frac{1}{3}\left[\left(1 - \int_0^t g(s)\,\mathrm{d}s\right)\|\Delta u\|_2^2 + \|\Delta u_t\|_2^2 + go\Delta u\right] + \frac{k}{4}\|u\|_2^2 + \frac{1}{6}I(t). \tag{53}$$

Since $I(0) > 0$ and I is continuous on $[0, T]$, there exists $t_0 \in (0, T]$ such that $I(t) > 0$, for all $t \in [0, t_0)$. Let us denote by t_0 the largest real number in $(0, T]$ such that $I > 0$ on $[0, t_0)$. If $t_0 = T$, then (51) is satisfied.

We assume by contradiction that $t_0 \in (0, T)$. Thus $I(t_0) = 0$ and

$$\|\Delta u(t)\|_2^2 \le \frac{6}{\ell}J(t) \le \frac{6}{\ell}E(t) \le \frac{6}{\ell}E(0), \quad \forall t \in [0, t_0). \tag{54}$$

If $\|\Delta u(t_0)\|_2^2 = 0$, then (48) and (49) give

$$\begin{aligned}0 = I(t_0) &= \left(1 - \int_0^{t_0} g(s)\,\mathrm{d}s\right)\|\Delta u(t_0)\|_2^2 + \|\Delta u_t(t_0)\|_2^2 + go\Delta u(t_0)\\ &\quad - 3k\int_{\Omega} u^2(t_0)\ln|u(t_0)|\,\mathrm{d}x\\ &\le c\|\Delta u(t_0)\|_2^2 + go\Delta u(t_0)\\ &= \int_0^{t_0} g(s)\|\Delta u(s)\|_2^2\,\mathrm{d}s.\end{aligned} \tag{55}$$

Consequently, if $g > 0$ on $[0, t_0)$, we get

$$\|\Delta u(s)\|_2 = 0, \quad \forall s \in [0, t_0).$$

Then

$$I(t) = 0, \quad \forall t \in [0, t_0),$$

which is not true since $I > 0$ on $[0, t_0)$. If $g \ne 0$ on $[0, t_0)$, then let $t_1 \in [0, t_0)$ the smallest real number such that $g(t_1) = 0$. Because $g(0) > 0$ and g is positive, nonincreasing and continuous on $\mathbb{R}^+$

(condition (A1)), then $t_1 > 0$ and $g = 0$ on $[t_1, \infty)$. Therefore, from (55), we deduce that

$$0 = \int_0^{t_0} g(s)\|\Delta u(s)\|_2^2\, ds = \int_0^{t_1} g(s)\|\Delta u(s)\|_2^2\, ds,$$

then $\|\Delta u(s)\|_2 = 0$, for any $s \in [0, t_1)$, which implies that $I(t) = 0$, for any $t \in [0, t_1)$. As in above, this is a contraduction with the fact that $I > 0$ on $[0, t_0)$. Then we conclude that $\|\Delta u(t_0)\|_2^2 > 0$. On the other hannd, we have

$$I(t_0) \geq \ell\|\Delta u(t_0)\|_2^2 - 3k\int_\Omega u(t_0)^2 \ln |u(t_0)|\, dx.$$

By using (54) and Lemma 4.1, we have

$$I(t_0) \geq \left[\ell - 3kc_*^3\left(\frac{6E(0)}{\ell}\right)^{1/2}\right]\|\Delta u(t_0)\|_2^2.$$

By recalling (50), we arrive at $I(t_0) > 0$, which contradicts the assumption $I(t_0) = 0$. Hence, $t_0 = T$ and then

$$I(t) > 0, \quad \forall t \in [0, T). \qquad \blacksquare$$

5. Stability

In this section, we state and prove our stability result. We start by establishing several lemmas needed for the proof of our main result.

Lemma 5.1: *Assume that g satisfies* (A1). *Then, for* $u \in H_0^2(\Omega)$, *we have*

$$\int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))\, ds\right)^2 dx \leq c(go\Delta u)(t)$$

and

$$\int_\Omega\left(\int_0^t g'(t-s)(u(t)-u(s))\, ds\right)^2 dx \leq -c(g'o\Delta u)(t).$$

Proof:

$$\int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))\, ds\right)^2 dx = \int_\Omega\left(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}(u(t)-u(s))\, ds\right)^2 dx.$$

By applying Cauchy-Schwarz' and Poincaré's inequalities, we can show that

$$\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-s)(u(t)-u(s))\, ds\right)^2 dx \\
&\leq \int_\Omega\left(\int_0^t g(t-s)\, ds\right)\left(\int_0^t g(t-s)(u(t)-u(s))^2\, ds\right) dx \\
&\leq (1-\ell)c(go\Delta u)(t) \\
&\leq c(go\Delta u)(t). \qquad (56)
\end{aligned}$$

Similarly, the second inequality in Lemma 5.1 can be proved. $\blacksquare$

Lemma 5.2: *Assume that g satisfies* (A1) *and* (A2). *Then*

$$\int_0^\infty \xi(t)g^{1-\sigma}(t)\,\mathrm{d}t < \infty, \quad \forall\sigma < 2-p. \tag{57}$$

Proof: Using (A1) and (A2), we easily see that, for any $\sigma < 2-p$,

$$\xi(t)g^{1-\sigma}(t) = \xi(t)g^{1-\sigma}(t)g^p(t)g^{-p}(t) \le -g'(t)g^{1-\sigma-p}(t).$$

Integrate the last inequality over $(0,\infty)$, we obtain

$$\int_0^\infty \xi(t)g^{1-\sigma}(t)\,\mathrm{d}t \le -\int_0^\infty g'(t)g^{1-\sigma-p}(t)\,\mathrm{d}t = \left[-\frac{g^{2-p-\sigma}(t)}{2-p-\sigma}\right]_0^\infty < \infty. \quad \blacksquare$$

Similar to Cavalcanti and Oquendo [38], we can easily have the following lemma:

Lemma 5.3: *Assume that* (A1)–(A3) *and (50) hold and u is a solution of (1). Then, for any* $0<\sigma<1$, *we have*

$$(go\Delta u)(t) \le c\left[\left(\int_0^t g^{1-\sigma}(t)\,\mathrm{d}t\right)E(0)\right]^{(p-1)/(p-1+\sigma)} (g^p o\Delta u)^{\sigma/(p-1+\sigma)}.$$

By taking $\sigma = \frac{1}{2}$, *we get*

$$(go\Delta u)(t) \le c\left(\int_0^t g^{1/2}(s)\,\mathrm{d}s\right)^{(2p-2)/(2p-1)} (g^p o\Delta u)^{1/(2p-1)}(t) \tag{58}$$

and, for any $\epsilon_0 \in (0,1)$,

$$(go\Delta u)^{1/(1+\epsilon_0)}(t) \le c^{1/(1+\epsilon_0)}\left(\int_0^t g^{1/2}(s)\,\mathrm{d}s\right)^{(2p-2)/((2p-1)(1+\epsilon_0))} (g^p o\Delta u)^{1/((2p-1)(1+\epsilon_0))}(t). \tag{59}$$

Corollary 5.4: *Assume that* (A1)–(A3) *and (50) hold and u is a solution of (1). Then*

$$\xi(t)(go\Delta u)(t) \le c(-E'(t))^{1/(2p-1)} \tag{60}$$

and, for any $\epsilon_0 \in (0,1)$,

$$\xi(t)(go\Delta u)^{1/(1+\epsilon_0)}(t) \le c_{\epsilon_0} - E'(t))^{1/((2p-1)(1+\epsilon_0))}. \tag{61}$$

Proof: Multiply both sides of (58) by $\xi(t)$ and use (57) and (14) to obtain

$$\begin{aligned}
\xi(t)(go\Delta u)(t) &\le c\xi^{(2p-2)/(2p-1)}(t)\left(\int_0^t g^{1/2}(s)\,\mathrm{d}s\right)^{(2p-2)/(2p-1)} \xi^{1/(2p-1)}(t)(g^p o\Delta u)^{1/(2p-1)}(t)\\
&\le c\left(\int_0^t \xi(s)g^{1/2}(s)\,\mathrm{d}s\right)^{(2p-2)/(2p-1)} (\xi g^p o\Delta u)^{1/(2p-1)}(t)\\
&\le c\left(\int_0^\infty \xi(s)g^{1/2}(s)\,\mathrm{d}s\right)^{(2p-2)/(2p-1)} (-g' o\Delta u)^{1/(2p-1)}(t)\\
&\le c - E'(t))^{1/(2p-1)}.
\end{aligned} \tag{62}$$

For the proof of (61), using (60) and because ξ is nonnegative and nonincreasing, we obtain

$$\xi(t)(go\Delta u)^{1/(1+\epsilon_0)}(t) = \xi^{\epsilon_0/(1+\epsilon_0)}(t)(\xi(t)(go\Delta u)(t))^{1/(1+\epsilon_0)} \le c_{\epsilon_0} - E'(t))^{1/((2p-1)(1+\epsilon_0))}. \quad \blacksquare$$

Lemma 5.5: *Under the assumptions* (A1)–(A3) *and (50), the functional*

$$\psi(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u \,\mathrm{d}x + \int_\Omega \Delta u \Delta u_t \,\mathrm{d}x$$

satisfies, along the solution of (1), the estimate

$$\begin{aligned}\psi'(t) \le & -\frac{\ell}{2}\int_\Omega |\Delta u|^2 \,\mathrm{d}x + \int_\Omega |\Delta u_t|^2 \,\mathrm{d}x + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} \,\mathrm{d}x + c(go\Delta u)(t) \\ & + k\int_\Omega u^2 \ln|u| \,\mathrm{d}x. \end{aligned} \tag{63}$$

Proof: Direct differentiation of ψ, using (1), yields

$$\begin{aligned}\psi'(t) = & -\int_\Omega |\Delta u|^2 \,\mathrm{d}x + \int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\,\mathrm{d}x \\ & + \int_\Omega |\Delta u_t|^2 \,\mathrm{d}x + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} \,\mathrm{d}x + k\int_\Omega u^2 \ln|u| \,\mathrm{d}x. \end{aligned} \tag{64}$$

We then estimate the second term on the right side of (64). We have, using (10),

$$\begin{aligned}\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\,\mathrm{d}x &= \int_\Omega \Delta u(t)\int_0^t g(t-s)(\Delta u(s)-\Delta u(t)+\Delta u(t))\,\mathrm{d}s\,\mathrm{d}x \\ &\le (1-\ell)\int_\Omega |\Delta u|^2 \,\mathrm{d}x - \int_\Omega \left(\int_0^t g(t-s)\,(\Delta u(t)-\Delta u(s))\,\mathrm{d}s\right)\mathrm{d}x. \end{aligned}$$

By exploiting Lemma 5.1 and

$$ab \le \frac{1}{2\eta}a^2 + \frac{\eta}{2}b^2, \quad \forall a,b \ge 0,\ \forall \eta > 0,$$

we arrive at

$$\begin{aligned}&\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad\le (1-\ell)\int_\Omega |\Delta u|^2 \,\mathrm{d}x + \frac{1}{2\eta}\int_\Omega \left(\int_0^t g(t-s)|\Delta u(t)-\Delta u(s)|\,\mathrm{d}s\right)^2 \mathrm{d}x \\ &\qquad + \frac{\eta}{2}\int_\Omega |\Delta u|^2 \,\mathrm{d}x \\ &\quad\le \left(1-\ell+\frac{\eta}{2}\right)\int_\Omega |\Delta u|^2 \,\mathrm{d}x + \frac{c}{\eta}(go\Delta u)(t). \end{aligned}$$

By taking $\eta = \ell$, we find

$$\int_\Omega \Delta u(t)\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\,\mathrm{d}x \le \frac{2-\ell}{2}\int_\Omega |\Delta u|^2 \,\mathrm{d}x + c(go\Delta u)(t). \tag{65}$$

Inserting (65) in (64), estimate (63) is established. ■

Lemma 5.6: *Under the assumptions* (A1)–(A3) *and (50), the functional*

$$\chi(t) := -\int_\Omega \left(\Delta^2 u_t + \frac{1}{\rho+1}|u_t|^\rho u_t\right) \int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\,\mathrm{d}x$$

satisfies, along the solution of (1) and for any $\delta, \delta_1, \delta_2 > 0$, *the estimate*

$$\begin{aligned}\chi'(t) \le & \left[(1+2(1-\ell)^2)\delta_1 + \frac{\delta}{4}\right]\int_\Omega |\Delta u|^2\,\mathrm{d}x - \frac{1}{\rho+1}\left(\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega |u_t|^{\rho+2}\,\mathrm{d}x \\ & + c\left(\delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right)(go\Delta u)(t) - \frac{c}{\delta_2}(g' o\nabla u)(t) \\ & + \left[\delta_2 + c\delta_2(E(0))^\rho - \int_0^t g(s)\,\mathrm{d}s\right]\int_\Omega |\Delta u_t|^2\,\mathrm{d}x + c_{\epsilon_0,\delta}(go\Delta u)^{1/(1+\epsilon_0)}(t). \qquad (66)\end{aligned}$$

Proof: Differentiating χ with respect to t and making use of (1), we find

$$\begin{aligned}\chi'(t) = & \int_\Omega \Delta u(t) \int_0^t g(t-s)(\Delta u(s) - \Delta u(t))\,\mathrm{d}s\,\mathrm{d}x \\ & - \int_\Omega \left(\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\right)\left(\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))\,\mathrm{d}s\right)\mathrm{d}x \\ & - \left(\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega |\Delta u_t|^2\,\mathrm{d}x - \int_\Omega \Delta u_t(t)\int_0^t g'(s)(\Delta u(s)-\Delta u(t))\,\mathrm{d}s\,\mathrm{d}x \\ & - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)(u(s)-u(t))\,\mathrm{d}s\,\mathrm{d}x \\ & - \frac{1}{\rho+1}\left(\int_0^t g(s)\,\mathrm{d}s\right)\int_\Omega |u_t|^{\rho+2}\,\mathrm{d}x - k\int_\Omega u\ln|u| \int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\,\mathrm{d}x. \qquad (67)\end{aligned}$$

Now we proceed, using repeatedly Cauchy-Schwarz' inequality, Young's inequality and Lemma 5.1, to estimate each term in the right-hand side of (67). The first term may be estimated as follows

$$\begin{aligned}& \int_\Omega \Delta u(t)\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))\,\mathrm{d}s\,\mathrm{d}x \\ & \quad \le \delta_1 \int_\Omega |\Delta u|^2\,\mathrm{d}x + \frac{c}{\delta_1}(go\Delta u)(t), \quad \forall \delta_1 > 0. \qquad (68)\end{aligned}$$

For the second term, we recall (10) and the fact that $(a+b)^2 \le 2(a^2+b^2)$ to get, for any $\delta_1 > 0$,

$$\begin{aligned}& -\int_\Omega \left(\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\right)\left(\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))\,\mathrm{d}s\right)\mathrm{d}x \\ & \le \delta_1 \int_\Omega \left|\int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s\right|^2\mathrm{d}x + \frac{1}{4\delta_1}\int_\Omega \left|\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))\,\mathrm{d}s\right|^2\mathrm{d}x \\ & \le \delta_1 \int_\Omega \left(\int_0^t g(t-s)(|\Delta u(s)-\Delta u(t)| + |\Delta u(t)|)\,\mathrm{d}s\right)^2\mathrm{d}x + \frac{c}{\delta_1}(go\Delta u)(t) \\ & \le c\left(\delta_1 + \frac{1}{\delta_1}\right)(go\Delta u)(t) + 2\delta_1(1-\ell)^2\int_\Omega |\Delta u|^2\,\mathrm{d}x. \qquad (69)\end{aligned}$$

For the fourth term, it is easy to see that, for any $\delta_2 > 0$,

$$
\begin{aligned}
&-\int_\Omega \Delta u_t \int_0^t g'(t-s)(\Delta u(s)-\Delta u(t))\,\mathrm{d}s\,\mathrm{d}x \\
&\le \delta_2 \int_\Omega |\Delta u_t|^2\,\mathrm{d}x + \frac{c}{\delta_2}\int_\Omega\int_0^t (-g'(t-s))|\Delta u(s)-\Delta u(t)|^2\,\mathrm{d}s\,\mathrm{d}x. \qquad (70)
\end{aligned}
$$

The fifth term may be handled similarly

$$
\begin{aligned}
&-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)(u(s)-u(t))\,\mathrm{d}s\,\mathrm{d}x \\
&\le \frac{1}{\rho+1}\left[\delta_2\int_\Omega |u_t|^{2(\rho+1)}\,\mathrm{d}x + \frac{c}{\delta_2}\int_\Omega\int_0^t \left(-g'(t-s)\right)|\Delta u(s)-\Delta u(t)|^2\,\mathrm{d}s\,\mathrm{d}x\right]. \qquad (71)
\end{aligned}
$$

Using (13), (14), (46), (51) and (53), we have

$$
E(0) \ge E(t) = J(t) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} \ge J(t) \ge \frac{1}{6}\|\Delta u_t\|_2^2,
$$

which gives

$$
\|\Delta u_t\|_2^2 \le 6E(0). \qquad (72)
$$

By exploiting the Sobolev embedding

$$
H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega), \qquad (73)
$$

and (72), we obtain

$$
\int_\Omega |u_t|^{2(\rho+1)}\,\mathrm{d}x \le c(E(0))^\rho \|\Delta u_t\|_2^2. \qquad (74)
$$

Therefore (71) takes the form

$$
\begin{aligned}
&-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^{+\infty} g'(s)(u(t-s)-u(t))\,\mathrm{d}s\,\mathrm{d}x \\
&\le c\delta_2(E(0))^\rho\|\Delta u_t\|_2^2 - \frac{c}{\delta_2}(g' o \Delta u)(t). \qquad (75)
\end{aligned}
$$

Applying (19) for $s=|u|$, using the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ and performing the same calulactions as before, we get, for any $\delta_3 > 0$ and any $\varepsilon_0 \in (0,1)$,

$$
\begin{aligned}
&-k\int_\Omega u\ln|u| \int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\,\mathrm{d}x \\
&\le k\int_\Omega (u^2 + d_{\epsilon_0}|u|^{1-\epsilon_0})\left|\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\,\mathrm{d}x\right| \\
&\le c\int_\Omega |u|^2\left|\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\right|\,\mathrm{d}x + \delta_3\int_\Omega u^2\,\mathrm{d}x \\
&\quad + c_{\epsilon_0,\delta_3}\int_\Omega \left|\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\right|^{2/(1+\epsilon_0)}\mathrm{d}x \\
&\le c\delta_3||\Delta u||_2^2 + \frac{c}{\delta_3}\int_\Omega\left|\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\right|^2\mathrm{d}x \\
&\quad + c_{\epsilon_0,\delta_3}\int_\Omega \left|\int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\right|^{2/(1+\epsilon_0)}\mathrm{d}x,
\end{aligned}
$$

then, puting $\delta/4 = c\delta_3$ and using Holder's inequality and Lemma 5.1, we find

$$-k\int_{\Omega} u\ln|u| \int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\,\mathrm{d}x \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(go\Delta u)(t) + c_{\epsilon_0,\delta}(go\Delta u)^{1/(1+\epsilon_0)}(t). \tag{76}$$

Combining (67)–(70), (75) and (76), estimate (66) is established. ■

Lemma 5.7: *Assume that* (A1)–(A3) *and (50) hold and let* $\varepsilon_0 \in (0,1)$. *Assume that*

$$0 < E(0) < \frac{e\ell\pi}{4c_p}. \tag{77}$$

Then, for k small enough, there exist positive constants ε *and N such that the functional*

$$L = NE + \varepsilon\psi + \chi$$

satisfies

$$L \sim E \tag{78}$$

and, for any $t_0 > 0$, *there exists a positive constant m such that*

$$L'(t) \leq -mE(t) + c(go\Delta u)(t) + c_{\epsilon_0}(go\Delta u)^{1/(1+\epsilon_0)}(t), \quad \forall t \geq t_0. \tag{79}$$

Proof: To prove (78), we use Young's inequality, the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$, (51), (53), (72) and (74) to obtain

$$\begin{aligned}|L(t) - NE(t)| \leq{}& \frac{\varepsilon}{\rho+2}||u_t||_{\rho+2}^{\rho+2} + \frac{\varepsilon}{(\rho+1)(\rho+2)}||u||_{\rho+2}^{\rho+2} + \frac{\varepsilon}{2}||\Delta u_t||_2^2 \\ &+ \frac{\varepsilon}{2}||\Delta u||_2^2 + \frac{1}{2(\rho+1)}||u_t||_{2(\rho+1)}^{2(\rho+1)} + \frac{1-\ell}{2(\rho+1)}c_p(go\Delta u)(t) \\ &+ \frac{1}{2}||\Delta u_t||_2^2 + \frac{1-\ell}{2}(go\Delta u)(t) \\ \leq{}& \varepsilon E(t) + \varepsilon\frac{c^{\rho+2}}{(\rho+1)(\rho+2)}\left(\frac{6}{\ell}E(0)\right)^{1+(\rho/2)}E(t) + 3\varepsilon E(t) \\ &+ \frac{3\varepsilon}{\ell}E(t) + \frac{3c}{(\rho+1)}(E(0))^{\rho}E(t) + \frac{3(1-\ell)}{(\rho+1)}c_pE(t) \\ &+ 3E(t) + 3(1-\ell)E(t) \\ \leq{}& c(1+\varepsilon)E(t),\end{aligned} \tag{80}$$

that is

$$[N - c(1+\varepsilon)]E(t) \leq L(t) \leq [N + c(1+\varepsilon)]E(t).$$

By fixing N large enough so that $N > c(1+\varepsilon)$, we obtain the desired result (78).

For the proof of (79), since g is positive and $g(0) > 0$ then, for any $t_0 > 0$, we have

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds = g_0 > 0, \quad \forall t \geq t_0.$$

By using (14), (63) and (66) then, for $t \geq t_0$, we have

$$\begin{aligned} L'(t) \leq & \left(\frac{N}{2} - \frac{c}{\delta_2}\right)(g' o \Delta u)(t) - \frac{g_0 - \varepsilon}{\rho + 1}\int_\Omega |u_t|^{\rho+2}\,dx \\ & - \left[\varepsilon\frac{\ell}{2} - (1 + 2(1-\ell)^2)\delta_1 - \frac{\delta}{4}\right]\|\Delta u\|_2^2 \\ & - [g_0 - \varepsilon - \delta_2 - c\delta_2(E(0))^\rho]\|\Delta u_t\|_2^2 \\ & + c\left(\varepsilon + \delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right)(go\Delta u)(t) \\ & + c_{\epsilon_0,\delta}(go\Delta u)^{1/(1+\epsilon_0)}(t) + \varepsilon k\int_\Omega u^2 \ln|u|\,dx. \end{aligned} \tag{81}$$

Using the definition of $E(t)$, we obtain, for any $m > 0$,

$$\begin{aligned} L'(t) \leq & -mE(t) + \left(\frac{N}{2} - \frac{c}{\delta_2}\right)(g' o \Delta u)(t) - \left(\frac{g_0 - \varepsilon}{\rho + 1} - \frac{m}{\rho+2}\right)\int_\Omega |u_t|^{\rho+2}\,dx \\ & - \left[\varepsilon\frac{\ell}{2} - (1 + 2(1-\ell)^2)\delta_1 - \frac{\delta}{4} - \frac{m(1-g_0)}{2}\right]\|\Delta u\|_2^2 \\ & - \left[g_0 - \varepsilon - \delta_2 - c\delta_2(E(0))^\rho - \frac{m}{2}\right]\|\Delta u_t\|_2^2 \\ & + \left[c\left(\varepsilon + \delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right) + \frac{m}{2}\right](go\Delta u)(t) \\ & + c_{\epsilon_0,\delta}(go\Delta u)^{1/(1+\epsilon_0)}(t) + \frac{mk}{4}\|u\|_2^2 \\ & + \left(\varepsilon - \frac{m}{2}\right)k\int_\Omega u^2 \ln|u|\,dx. \end{aligned} \tag{82}$$

Using the Logarithmic Sobolev inequality (16), we get, for $0 < m < 2\varepsilon$,

$$\begin{aligned} L'(t) \leq & -mE(t) + \left[\frac{N}{2} - \frac{c}{\delta_2}\right](g' o \Delta u)(t) - \left(\frac{g_0 - \varepsilon}{\rho + 1} - \frac{m}{\rho+2}\right)\int_\Omega |u_t|^{\rho+2}\,dx \\ & - \left[\varepsilon\frac{\ell}{2} - (1 + 2(1-\ell)^2)\delta_1 - \frac{\delta}{4} - \frac{m(1-g_0)}{2} - \left(\varepsilon - \frac{m}{2}\right)\frac{kc_p a^2}{2\pi}\right]\|\Delta u\|_2^2 \\ & - \left(g_0 - \varepsilon - \delta_2 - c\delta_2(E(0))^\rho - \frac{m}{2}\right)\|\Delta u_t\|_2^2 \\ & + \left[c\left(\varepsilon + \delta_1 + \frac{1}{\delta_1} + \frac{1}{\delta}\right) + \frac{m}{2}\right](go\Delta u)(t) + c_{\epsilon_0,\delta}(go\Delta u)^{1/(1+\epsilon_0)}(t) \\ & - \left(\varepsilon - \frac{m}{2}\right)\frac{k}{2}(2(1 + \ln a) - \ln\|u\|_2^2)\|u\|_2^2 + \frac{mk}{4}\|u\|_2^2. \end{aligned} \tag{83}$$

At this point, we choose our constant carefully. First, we pick $0 < \varepsilon < g_0$, then δ_1, δ_2 and δ small enough so that

$$k_1 := \varepsilon\frac{\ell}{2} - (1 + 2(1-\ell)^2)\delta_1 - \frac{\delta}{4} > 0$$

and

$$k_2 := g_0 - \varepsilon - \delta_2 - c\delta_2(E(0))^{\rho} > 0.$$

Then, N sufficiently large so that

$$N > c(1+\varepsilon) \quad \text{and} \quad \frac{N}{2} - \frac{c}{\delta_2} \geq 0.$$

Consequently, we get

$$\begin{aligned} L'(t) \leq & -mE(t) - \left(\frac{g_0 - \varepsilon}{\rho + 1} - \frac{m}{\rho + 2}\right) \int_{\Omega} |u_t|^{\rho+2}\,\mathrm{d}x \\ & - \left[k_1 - \frac{m(1-g_0)}{2} - \left(\varepsilon - \frac{m}{2}\right)\frac{kc_p a^2}{2\pi}\right] \|\Delta u\|_2^2 \\ & - \left(k_2 - \frac{m}{2}\right)\|\Delta u_t\|_2^2 + \left(c + \frac{m}{2}\right)(go\Delta u)(t) \\ & + c_{\epsilon_0}(go\Delta u)^{1/(1+\epsilon_0)}(t) + \frac{mk}{4}\|u\|_2^2 \\ & - \left(\varepsilon - \frac{m}{2}\right)\frac{k}{2}(2(1+\ln a) - \ln \|u\|_2^2)\|u\|_2^2. \end{aligned} \tag{84}$$

Finally, we choose m and k small enough so that $m \leq \varepsilon$ (so $(mk)/4 \leq (\varepsilon - m/2)k/2$),

$$\frac{g_0 - \varepsilon}{\rho + 1} - \frac{m}{\rho + 2} > 0,$$

$$k_1 - \frac{m(1-g_0)}{2} - \left(\varepsilon - \frac{m}{2}\right)\frac{kc_p a^2}{2\pi} > 0$$

and

$$k_2 - \frac{m}{2} > 0,$$

we get

$$\begin{aligned} L'(t) \leq & -mE(t) + c(go\Delta u)(t) + c_{\varepsilon_0}(go\Delta u)^{1/(1+\varepsilon_0)}(t) \\ & - \left(\varepsilon - \frac{m}{2}\right)\frac{k}{2}(1 + 2\ln a - \ln \|u\|_2^2)\|u\|_2^2. \end{aligned} \tag{85}$$

Using (13), (14), (46), (51), (53) and (77), we have

$$\ln \|u\|_2^2 \leq \ln\left(\frac{4}{k}J(t)\right) \leq \ln\left(\frac{4}{k}E(t)\right) \leq \ln\left(\frac{4}{k}E(0)\right) \leq \ln\left(\frac{e\ell\pi}{kc_p}\right). \tag{86}$$

By taking a satisfying

$$\max\left\{\mathrm{e}^{-3/2}, \sqrt{\frac{\ell\pi}{kc_p}}\right\} < a < \sqrt{\frac{2\ell\pi}{kc_p}}$$

(so (26) is satisfied), we guarantee

$$1 + 2\ln a - \ln \|u\|_2^2 \geq 0.$$

Which completes the proof of (79). ∎

Remark 5.8: Using (10), (13), (46), (51) and (53), we have

$$E(t) = J(t) + \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} \geq J(t) \geq \frac{l}{6}\|\Delta u(t)\|_2^2,$$

then, using (14),

$$\|\Delta u(t)\|_2^2 \leq \frac{6}{l}E(t) \leq \frac{6}{l}E(0). \tag{87}$$

So, from (14) and using Young's inequality, we get

$$\begin{aligned} |E'(t)| &= \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \frac{1}{2}(g' o \Delta u)(t) \\ &\leq \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \int_0^t g'(t-s)(\|\Delta u(t)\|_2^2 + \|\Delta u(s)\|_2^2)\,\mathrm{d}s \\ &\leq \frac{6}{l}\left(\frac{1}{2}g(t) + 2g(0) - 2g(t)\right)E(0) \\ &\leq cE(0). \end{aligned} \tag{88}$$

Theorem 5.9: *Let* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$, $\epsilon \in (0, 2p-1)$ *and* $t_0 > 0$. *Assume that* (A1)–(A3) *and (50) hold. Then, for k small enough, there exists a positive constant K such that the solution of (1) satisfies*

$$E(t) \leq K\left(1 + \int_{t_0}^t \xi^{2p-1+\epsilon}(s)\,\mathrm{d}s\right)^{-1/(2p-2+\epsilon)}, \quad \forall t \geq t_0. \tag{89}$$

Moreover, if there exist $\epsilon_1 \in (0, 2p-1)$ *and* $t_0 > 0$ *such that*

$$\int_{t_0}^{\infty}\left(1 + \int_{t_0}^t \xi^{2p-1+\epsilon_1}(s)\,\mathrm{d}s\right)^{-1/(2p-2+\epsilon_1)} \mathrm{d}t < \infty, \tag{90}$$

then, for any $r \in (0, p)$ *and* $t_0 > 0$, *there exists a positive constant K such that the solution of (1) satisfies*

$$E(t) \leq K\left(1 + \int_{t_0}^t \xi^{p+r}(s)\,\mathrm{d}s\right)^{-1/(p-1+r)}, \quad \forall t \geq t_0. \tag{91}$$

Remark 5.10: Using (89) and (90), we can easily show that

$$\int_0^{+\infty} E(t)\,\mathrm{d}t < +\infty. \tag{92}$$

Proof: We multiply (79) by $\xi(t)$ and use Corollary 5.4 and (88) to get, for any $t \geq t_0$,

$$\begin{aligned} \xi(t)L'(t) &\leq -m\xi(t)E(t) + c(-E'(t))^{1/(2p-1)} + c(-E'(t))^{1/((2p-1)(1+\epsilon_0))} \\ &\leq -m\xi(t)E(t) + c(-E'(t))^{\epsilon_0/((2p-1)(1+\epsilon_0))}(-E'(t))^{1/((2p-1)(1+\epsilon_0))} \\ &\quad + c(-E'(t))^{1/((2p-1)(1+\epsilon_0))} \\ &\leq -m\xi(t)E(t) + c(-E'(t))^{1/((2p-1)(1+\epsilon_0))}, \quad \forall t \geq t_0. \end{aligned} \tag{93}$$

Multiply the last inequality by $\xi^{\gamma}(t)E^{\gamma}(t)$, where $\gamma = (2p-1)(1+\epsilon_0) - 1$, and notice that $\xi' \le 0$ to obtain

$$\xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) \le -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c(\xi E)^{\gamma}(t)(-E'(t))^{1/(\gamma+1)}, \quad \forall t \ge t_0.$$

Use of Young's inequality, with $q = \gamma + 1$ and $q^* = (\gamma+1)/\gamma$, gives, for any $\varepsilon' > 0$,

$$\begin{aligned}\xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) &\le -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c(\varepsilon'\xi^{\gamma+1}(t)E^{\gamma+1} - c_{\varepsilon'}E'(t)) \\ &= -(m-\varepsilon' c)\xi^{\gamma+1}(t)E^{\gamma+1} - cE'(t), \quad \forall t \ge t_0.\end{aligned}$$

We then choose $0 < \varepsilon' < m/c$ and recall that $\xi' \le 0$ and $E' \le 0$, to get, for $c_1 = m - \varepsilon' c$,

$$(\xi^{\gamma+1}E^{\gamma}L)'(t) \le \xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) \le -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t) - cE'(t), \quad \forall t \ge t_0,$$

which implies

$$(\xi^{\gamma+1}E^{\gamma}L + cE)'(t) \le -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t), \quad \forall t \ge t_0.$$

Let $F = \xi^{\gamma+1}E^{\gamma}L + cE$. Then $F \sim E$ (thanks to (78)) and

$$F'(t) \le -c\xi^{\gamma+1}(t)F^{\gamma+1}(t) = -c\xi^{(2p-1)(1+\epsilon_0)}(t)F^{(2p-1)(1+\epsilon_0)}(t), \quad \forall t \ge t_0.$$

Integrating over (t_0, t) and using the fact that $F \sim E$, we obtain (89) with $\epsilon = (2p-1)\epsilon_0$.

To establish (91), we use the idea of Messaoudi and Al-Khulaifi [12]. Let

$$\eta(t) = \int_0^t \|\Delta u(t) - \Delta u(t-s)\|_2^2 \,\mathrm{d}s.$$

Using (89), (87), (90) and (92), we have

$$\begin{aligned}\eta(t) &\le 2\int_0^t (\|\Delta u(t)\|_2^2 + \|\Delta u(t-s)\|_2^2)\,\mathrm{d}s \\ &\le \frac{12}{l}\int_0^t (E(t) + E(t-s))\,\mathrm{d}s \\ &\le \frac{24}{l}\int_0^t E(s)\,\mathrm{d}s < \frac{24}{l}\int_0^\infty E(s)\,\mathrm{d}s < \infty.\end{aligned}$$

This implies that

$$\sup_{t>0} \eta^{1-(1/p)}(t) < \infty. \tag{94}$$

Assume that $\eta(t) > 0$. Then, because ξ is nonincreasing, we find

$$\xi(t)(g \circ \Delta u)(t) \le \frac{\eta(t)}{\eta(t)}\int_0^t (\xi^p(s)g^p(s))^{1/p}\|\Delta u(t) - \Delta u(t-s)\|_2^2 \,\mathrm{d}s.$$

Applying Jensen's inequality to get

$$\xi(t)(g \circ \Delta u)(t) \le \eta(t)\left(\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\Delta u(t) - \Delta u(t-s)\|_2^2, \mathrm{d}s\right)^{1/p}.$$

Therefore, using (A2) and (94) we obtain

$$\begin{aligned}\xi(t)(g \circ \Delta u)(t) &\le \eta^{1-(1/p)}(t)\left(\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\Delta u(t) - \Delta u(t-s)\|_2^2 \,\mathrm{d}s\right)^{1/p} \\ &\le c(-g' \circ \Delta u)^{1/p}(t),\end{aligned}$$

and then, according to (14),

$$\xi(t)(g \circ \Delta u)(t) \leq c(-E'(t))^{1/p}. \tag{95}$$

So, since ξ is nonincreasing,

$$\begin{aligned}\xi(t)(g \circ \Delta u)^{1/(1+\epsilon_0)}(t) &= (\xi^{\epsilon_0}(t)\xi(t)(g \circ \Delta u)(t))^{1/(1+\epsilon_0)} \\ &\leq (\xi^{\epsilon_0}(0)\xi(t)(g \circ \Delta u)(t))^{1/(1+\epsilon_0)} \\ &\leq c(\xi(t)(g \circ \Delta u)(t))^{1/(1+\epsilon_0)} \\ &\leq c(-E'(t))^{1/(p(1+\epsilon_0))}. \end{aligned} \tag{96}$$

If $\eta(t) = 0$, then $s \to \Delta u(s)$ is a constant function on $[0, t]$. Therefore

$$(g \circ \Delta u)(t) = 0,$$

and hence (95) and (96) hold.

Now, multiplying (79) by $\xi(t)$ and using (88), (95) and (96) to find, for any $t \geq t_0$ (as for (93)),

$$\begin{aligned}\xi(t)L'(t) &\leq -m\xi(t)E(t) + c(-E'(t))^{1/p} + c(-E'(t))^{1/(p(1+\epsilon_0))} \\ &\leq -m\xi(t)E(t) + c(-E'(t))^{\epsilon_0/(p(1+\epsilon_0))}(-E'(t))^{1/(p(1+\epsilon_0))} + c(-E'(t))^{1/(p(1+\epsilon_0))} \\ &\leq -m\xi(t)E(t) + c(-E'(t))^{1/(p(1+\epsilon_0))}, \quad \forall t \geq t_0. \end{aligned} \tag{97}$$

Inequality (93) with $2p-1$ replaced by p is exactely (97). Then, the proof of (91) can be completed as for the one of (89) (by taking $\gamma = p(1+\epsilon_0) - 1$ and $\epsilon = p\epsilon_0$). This completes the proof of our main result. ■

Remark 5.11: We note here that $2p - 2 + \epsilon$ and $p - 1 + \epsilon$ can be arbitrary close to $2p-2$ and $p-1$, respectively, since ϵ can be arbitrary close to zero. On the other hand, in the absence of the logarithmic 'forcing' term ($k=0$), the estimates (19) and (76) drop out and, consequently, (79) takes the form

$$L'(t) \leq -mE(t) + c(go\Delta u)(t), \quad \forall t \geq t_0. \tag{98}$$

In this case, we obtain the following result.

Theorem 5.12: *Let $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$ and $t_0 > 0$. Assume that* (A1)–(A2) *hold. Then, there exists a positive constant K such that the solution of (1) satisfies, for all $t \geq t_0$,*

$$E(t) \leq Ke^{-\lambda \int_{t_0}^t \xi(s)\,ds} \quad \text{if } p = 1 \tag{99}$$

and

$$E(t) \leq K\left(1 + \int_{t_0}^t \xi^{2p-1}(s)\,ds\right)^{-1/(2p-2)} \quad \text{if } 1 < p < \frac{3}{2}. \tag{100}$$

Moreover, if $1 < p < \frac{3}{2}$ and

$$\int_0^\infty \left(1 + \int_{t_0}^t \xi^{2p-1}(s)\,ds\right)^{-1/(2p-2)} dt < \infty, \tag{101}$$

then

$$E(t) \leq K\left(1 + \int_{t_0}^t \xi^{p}(s)\,ds\right)^{-1/(p-1)}, \quad \forall t \geq t_0. \tag{102}$$

Acknowledgements

The authors thank KFUPM for its continuous support.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was funded by King Fahd University of Petroleum and Minerals (KFUPM) under Project #IN151023. This work was initiated during the visit of the second author to KFUPM in December 2016.

References

[1] Cavalcanti MM, Domingos Cavalcanti VN, Ferreira J. Existence and uniform decay for nonlinear viscoelastic equation with strong damping. Math Methods Appl Sci. 2001;24:1043–1053.
[2] Messaoudi SA, Tatar N-E. Global existence asymptotic behavior for a non-linear viscoelastic problem. Math Methods Sci Res. 2003;7(4):136–149.
[3] Messaoudi SA, Tatar N-E. Global existence and uniform stability of solutions for a quasilinear viscoelastic problem. Math Methods Appl Sci. 2007;30:665–680.
[4] Messaoudi SA, Tatar N-E. Exponential and polynomial decay for a quasilinear viscoelastic equation. Nonlinear Anal TMA. 2008;68:785–793.
[5] Han X, Wang M. General decay of energy for a viscoelastic equation with nonlinear damping. Math Methods Appl Sci. 2009;32(3):346–358.
[6] Messaoudi SA. General decay of solution energy in a viscoelastic equation with a nonlinear source. Nonlinear Anal. 2008;69:2589–2598.
[7] Messaoudi SA. General decay of solutions of a viscoelastic equation. J Math Anal App. 2008;341:1457–1467.
[8] Liu W. General decay and blow up of solution for a quasilinear viscoelastic equation with a nonlinear source. Nonlinear Anal. 2010;73:1890–1904.
[9] Liu W. General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms. J Math Phys. 2009;50(11):113506.
[10] Messaoudi SA, Mustafa M. A general stability result for a quasilinear wave equation with memory. Nonlin Anal. 2013;14:1854–1864.
[11] Cavalcanti MM, Domingos Cavalcanti V, Lasiecka I, et al. Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. Adv Nonlinear Anal. 2016; Available from: doi:10.1515/anona-2016-0027.
[12] Messaoudi SA, Al-Khulaifi W. General and optimal decay for a quasilinear viscoelastic equation. Appl Math Lett. 2017;66:16–22.
[13] Lagnese J. Asymptotic energy estimates for Kirchhoff plates subject to weak viscoelastic damping. (International Series of Numerical Mathematics; vol. 91). Bassel: Birhauser, Verlag; 1989.
[14] Muñoz Rivera J, Lapa EC, Barreto R. Decay rates for viscoelastic paltes with memory. J Elast. 1996;44:61–87.
[15] Komornik V. On the nonlinear boundary stabilization of Kirchoff Plates. NoDEA. 1994;1:323–337.
[16] Messaoudi SA. Global existence and nonexistence in a system of Petrovsky. J Math Anal Appl. 2002;265(2): 296–308.
[17] Chen W, Zhou Y. Global nonexistence for a semilinear Petrovsky equation. Nonlinear Anal A. 2009;70(9): 3203–3208.
[18] Santos M, Junior F. A boundary condition with memory for Kirchoff plates equations. Appl Math Comput. 2004;148:475–496.
[19] Benaissa A, Guesmia A. Energy decay of solutions of a wave equation of ϕ-Laplacian type with a general weakly nolinear dissipation. Elec J Diff Equ. 2008;109:1–22.
[20] Guesmia A. Existence globale et stabilisation interne non linéaire d'un système de Petrovsky. Bull Belg Math Soc. 1998;5:583–594.
[21] Han X, Wang M. General decay estimate of energy for the second order evolution equations with memory. Act Appl Math. 2010;110:194–207.
[22] Lagnese J. Boundary stabilization of thin plates. Philadelphia: SIAM; 1989.
[23] Lasiecka I. Exponential decay rates for the solutions of Euler-Bernoulli equations with boundary dissipation occurring in the moments only. J Differ Equ. 1992;95:169–182.
[24] Barrow JD, Parsons P. Inflationary models with logarithmic potentials. Phys Rev D. 1995;52:5576–5587.
[25] Enqvist K, McDonald J. Q-balls and baryogenesis in the MSSM. Phys Lett B. 1998;425:309–321.

[26] Bartkowski K, Gorka P. One-dimensional Klein-Gordon equation with logarithmic nonlinearities. J Phys A. 2008;41(35):355201.
[27] Bialynicki-Birula I, Mycielski J. Wave equations with logarithmic nonlinearities. Bull Acad Polon Sci Ser Sci Math Astronom Phys. 1975;23(4):461–466.
[28] Gorka P. Logarithmic Klein-Gordon equation. Acta Phys Polon B. 2009;40(1):59–66.
[29] Bialynicki-Birula I, Mycielski J. Nonlinear wave mechanics. Ann Phys. 1976;100(1–2):62–93.
[30] Gorka P, Prado H, Reyes EG. Nonlinear equations with infinitely many derivatives. Complex Anal Oper Theory. 2011;5(1):313–323.
[31] Vladimirov VS. The equation of the *p*-adic open string for the scalar tachyon field. Izv Math. 2005;69(3):487–512.
[32] Cazenave T, Haraux A. Equations d'evolution avec non-linearite logarithmique. Ann Fac Sci Toulouse Math. 1980;2(1):21–51.
[33] Hiramatsu T, Kawasaki M, Takahashi F. Numerical study of Q-ball formation in gravity mediation. J Cosmol Astropart Phys. 2010;2010(6):008.
[34] Han X. Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics. Bull Korean Math Soc. 2013;50(1):275–283.
[35] Chen H, Luo P, Liu GW. Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. J Math Anal Appl. 2015;422:84–98.
[36] Gross L. Logarithmic Sobolev inequalities. Amer J Math. 1975;97(4):1061–1083.
[37] Lions J. Quelques methods de resolution des probléms aux limites non lineaires. Paris: Dunod Gauthier-Villars; 1969.
[38] Cavalcanti MM, Oquendo HP. Frictional versus viscoelastic damping in a semilinear wave equation. SIAM J Control Optim. 2003;42(4):1310–1324.
[39] Lasiecka I, Tataru D. Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping. Differ Integral Equ. 1993;6(6):507–533.