THE EFFECT OF THE HEAT CONDUCTION OF TYPES I AND III ON THE DECAY RATE OF THE BRESSE SYSTEM VIA THE VERTICAL DISPLACEMENT

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Abstract. In this paper, we study the energy decay for two one-dimensional thermoelastic Bresse-type systems in a bounded open interval under mixed homogeneous Dirichlet-Neumann boundary conditions and with two different kinds of dissipation working only on the vertical displacement and given by heat conduction of types I and III. The two systems are consisting of three wave equations (Bresse-type system) coupled, in a certain manner, with one heat equation (type I) or with one wave equation (type III). We prove that, independently of the values of the coefficients, these systems are not exponentially stable. Moreover, we show the polynomial stability for each system with a decay rate depending on the smoothness of the initial data. The proof is based on the semigroup theory and a combination of the energy method and the frequency domain approach. Our results complete our study [10] for the case of a dissipation generated by an infinite memory.

Keywords: Bresse system, heat conduction, Asymptotic behavior, Energy method, Frequency domain approach.

AMS Classification: 35B40, 35L45, 74H40, 93D20, 93D15.

1. Introduction

We are interested in this paper in the asymptotic behavior at infinity of the solutions to two coupled systems related to the Bresse model with two different types of dissipation given by heat conduction and working only on the vertical displacement (the first equation of Bresse system). The first system is the Bresse system with thermoelasticity of type I (classical thermoelasticity known also as the Fourier law)

$$(1.1)\qquad \begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x+\psi+l\,w\right)_x - lk_0\left(w_x-l\varphi\right)+\delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x+\psi+l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x-l\varphi\right)_x + lk\left(\varphi_x+\psi+l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_3\theta_t - \beta\theta_{xx} + \delta\varphi_{xt} = 0 & \text{in } (0,1)\times(0,\infty) \end{cases}$$

along with the initial data

$$(1.2)\qquad \begin{cases} \varphi\left(x,0\right)=\varphi_0\left(x\right),\, \varphi_t\left(x,0\right)=\varphi_1\left(x\right) & \text{in } (0,1)\,, \\ \psi\left(x,0\right)=\psi_0\left(x\right),\, \psi_t\left(x,0\right)=\psi_1\left(x\right) & \text{in } (0,1)\,, \\ w\left(x,0\right)=w_0\left(x\right),\, w_t\left(x,0\right)=w_1\left(x\right) & \text{in } (0,1)\,, \\ \theta\left(x,0\right)=\theta_0\left(x\right) & \text{in } (0,1) \end{cases}$$

and the mixed homogeneous Dirichlet-Neumann boundary conditions

$$(1.3)\qquad \begin{cases} \varphi\left(0,t\right)=\psi_x\left(0,t\right)=w_x\left(0,t\right)=\theta_x\left(0,t\right)=0 & \text{in } (0,\infty)\,, \\ \varphi_x\left(1,t\right)=\psi\left(1,t\right)=w\left(1,t\right)=\theta\left(1,t\right)=0 & \text{in } (0,\infty)\,. \end{cases}$$

The second system is the Bresse system with thermoelasticity of type III

$$\text{(1.4)} \qquad \begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) + \delta\theta_{xt} = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_{tt} - \beta\theta_{xx} - \gamma\theta_{xxt} + \delta\varphi_{xt} = 0 & \text{in } (0,1)\times(0,\infty) \end{cases}$$

along with (1.2), (1.3) and

$$\text{(1.5)} \qquad \theta_t\left(x,0\right) = \theta_1\left(x\right) \quad \text{in } (0,1),$$

where ρ_1, ρ_2, ρ_3, b, k, k_0, δ, β, γ and l are positive constants, w, φ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements, and θ denotes the temperature.

The Bresse-type system is known as the circular arch problem and is given by the following equations:

$$\rho_1\varphi_{tt} = Q_x + lN + F_1, \quad \rho_2\psi_{tt} = M_x - Q + F_2 \quad \text{and} \quad \rho_1 w_{tt} = N_x - lQ + F_3,$$

with

$$N = k_0(w_x - l\varphi), \quad Q = k(\varphi_x + lw + \psi) \quad \text{and} \quad M = b\psi_x,$$

where ρ_1, ρ_2, l, k, k_0 and b are positive physical constants, N, Q and M denote, respectively, the axial force, the shear force and the bending moment, and w, φ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements. Here

$$\rho_1 = \rho A, \quad \rho_2 = \rho I, \quad k_0 = EA, \quad k = k'GA, \quad b = EI \quad \text{and} \quad l = R^{-1}$$

such that ρ, E, G, k', A, I and R are positive constants and denote, respectively, the density, the modulus of elasticity, the shear modulus, the shear factor, the cross-sectional area, the second moment of area of the cross-section and the radius of curvature. Finally, F_1, F_2 and F_3 are external forces, which play the role of controls of the system.

In order to stabilize Bresse-type systems, various choices of controls F_j (linear or nonlinear dampings, finite or infinite memories, heat conduction of different types, boundary feedbacks, ...) have been used in the literature and several decay results have been established, where the decay rate of solutions depends on the controls F_j, the regularity of the initial data and the coefficients ρ_1, ρ_2, l, k, k_0 and b. It is worthnoting that the system considered by Bresse [3] is obtained by taking

$$(F_1, F_2, F_3) = (0, -\gamma\psi_t, 0),$$

with $\gamma > 0$. For more details in what concerns mathematical modeling of the thermoelasticity, we refer the readers to the works [4], [7], [8], [16] and [17].

The well-posedness and stability of Bresse-type systems has attracted the attention of many researchers in the last few years. Under different types of direct or indirect controls, various stability results have been obtained, depending on the nature and the number of controls, the regularity of the initial data and the values of the coefficients. Let us focus our attention on the stability of Bresse system with indirect controls via the coupling with other parabolic and/or hyperbolic equations, which is the subject of the present paper.

The authors of [19] considered the damped Bresse system via the coupling with two heat equations

$$\text{(1.6)} \qquad \begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) + l\delta\theta = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta q_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_3\theta_t - \theta_{xx} + \beta\left(w_x - l\varphi\right)_t = 0, \\ \rho_3 q_t - q_{xx} + \beta\psi_{xt} = 0 \end{cases}$$

on $(0,L)\times(0,\infty)$, where $L > 0$, with homogeneous Dirichlet or mixed Dirichlet-Neumann boundary conditions. They proved the exponential stability of (1.6) if

$$\text{(1.7)} \qquad k\rho_2 - b\rho_1 = k - k_0 = 0.$$

Otherwise, the polynomial stability of (1.6) was proved in [19] with decay rates depending on the regularity of the initial data.

In [6], the authors considered the coupled Bresse system with only one heat equation

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0, \\ \rho_3\theta_t - \theta_{xx} + (\beta\psi_t)_x = 0 \end{cases} \tag{1.8}$$

on $(0, L) \times (0, \infty)$ and proved that the exponential stability of (1.8) is equivalent to (1.7), but (1.8) is polynomially stable in general. The results of [6] were extended in [21] to the local dissipation case; that is δ and β are functions on x and vanish on some part of $(0, L)$.

The authors of [15] considered the following thermoelastic Bresse system (known as the Cattaneo law):

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0 \end{cases} \tag{1.9}$$

in $(0, 1) \times (0, \infty)$ and proved that (1.9) is exponentially stable if

$$k - k_0 = \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) - \frac{\tau\delta^2}{b} = 0 \quad \text{and} \quad l \text{ is small},$$

and (1.9) is not exponentially stable if

$$k \neq k_0 \quad \text{or} \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) \neq \frac{\tau\delta^2}{b}.$$

Moreover, when

$$k - k_0 = 0, \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) \neq \frac{\tau\delta^2}{b} \quad \text{and} \quad l \text{ is small},$$

the polynomial stability for (1.9) was also proved in [15].

Recently in [1], it was proved that the exponential stability of

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta w_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0 \end{cases} \tag{1.10}$$

in $(0, 1) \times (0, \infty)$ is equivalent to

$$k\rho_2 - b\rho_1 = (k - k_0)\left(\rho_3 - \frac{\rho_1}{\tau k}\right) - \delta^2 = 0 \tag{1.11}$$

and

$$l^2 \neq \frac{k_0\rho_2 + b\rho_1}{k_0\rho_2}\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{k\rho_1}{\rho_2\left(k + k_0\right)}, \quad \forall m \in \mathbb{Z}. \tag{1.12}$$

Moreover, the polynomial stability of (1.10) in general was also proved in [1]. Very similar results to the ones of [1] was obtained in [2] and [11], where the dissipation is generated via the longitudinal displacements ($F_1 = F_2 = 0$) by, respectively, a linear frictional damping ($F_3 = -\gamma w_t$) and a thermoelastic effect of type I or type III.

The author of the present paper studied in [10] the case where the Bresse system is controled only via its vertical displacement by an infinite memory; that is

$$F_2 = F_3 = 0 \quad \text{and} \quad F_1 = -\int_0^\infty g(s)\varphi_{xx}(x, t-s)ds,$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given funcion converging exponentially to zero at inifinity. The author of [10] showed that this case is deeply different in the sense that, independently on the coefficients and the kernel g, the exponential stability does not hold, but the system is still stable at least polynomially, where the decay rate of solutions depends only on the smoothness of the initial data. For more reading about the stability of Bresse-type systems with infinite memories, we refer to [5], [9], [12], [13] and the references therein.

We mention that (1.6), (1.8), (1.9) and (1.10) are consisting of coupled conservative three hyperbolic equations and one or two dissipative parabolic equations, so the stability of the overall system is preserved thanks to the dissipation generated by the parabolic equations. In particular, under some relationship between the coefficients, the exponential stability of the whole system holds. Moreover, we note that in (1.6), the three hyperbolic equations are damped by the dissipation from the two heat equations. However, in (1.8) and (1.9), only the second hyperbolic equation is damped by the dissipation from the parabolic equation satisfied by θ, and in (1.10), only the third hyperbolic equation is damped by the dissipation from the fourth one. For our systems (1.1) and (1.4), only the first hyperbolic equation is damped by the dissipation from one heat equation (type I) or one wave equation (type III).

Contrary to the systems (1.6), (1.8), (1.9) and (1.10), and as in [10], we prove that $(1.1)-(1.3)$ and $(1.2)-(1.5)$ are not exponentially stable whatever the coefficients are. Moreover, we show the polynomial stability of $(1.1)-(1.3)$ and $(1.2)-(1.5)$, where the decay rate depends only on the smoothness of the initial data.

The proof of the well-posedness is based on the semigroup theory. However, the non-exponential and polynomial stability results are proved using the energy method combining with the frequency domain approach.

The paper is organized as follows. In section 2, we prove the well-posedness of $(1.1)-(1.3)$ and $(1.2)-(1.5)$. In section 3, we show that $(1.1)-(1.3)$ and $(1.2)-(1.5)$ are not exponentially stable. Section 4 will be devoted to the proof of the polynomial stability of $(1.1)-(1.3)$ and $(1.2)-(1.5)$. Finally, we end our paper by some general comments and related issues in section 5.

2. The semigroup setting

In this section, we give an idea on the proof of the well-posedness of $(1.1)-(1.3)$ and $(1.2)-(1.5)$. We introduce the spaces

$$\begin{cases} H_*^1(0,1) = \left\{ f \in H^1(0,1) : f(0) = 0 \right\}, \\ \widetilde{H}_*^1(0,1) = \left\{ f \in H^1(0,1) : f(1) = 0 \right\}, \\ H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1), \\ \widetilde{H}_*^2(0,1) = H^2(0,1) \cap \widetilde{H}_*^1(0,1) \end{cases}$$

and the energy space

$$\mathcal{H} = \widetilde{\mathcal{H}} \times \begin{cases} L^2(0,1) & \text{in case}\,(1.1), \\ \widetilde{H}_*^1(0,1) \times L^2(0,1) & \text{in case}\,(1.4), \end{cases}$$

where

$$\widetilde{\mathcal{H}} = H_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1),$$

equipped with the inner product

$$\langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} = k \left\langle (\varphi_{1x} + \psi_1 + l\, w_1), (\varphi_{2x} + \psi_2 + l\, w_2) \right\rangle_{L^2(0,1)} + b \left\langle \psi_{1x}, \psi_{2x} \right\rangle_{L^2(0,1)}$$

$$+ k_0 \left\langle (w_{1x} - l\varphi_1), (w_{2x} - l\varphi_2) \right\rangle_{L^2(0,1)} + \rho_1 \left\langle \tilde{\varphi}_1, \tilde{\varphi}_2 \right\rangle_{L^2(0,1)} + \rho_2 \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle_{L^2(0,1)} + \rho_1 \left\langle \tilde{w}_1, \tilde{w}_2 \right\rangle_{L^2(0,1)}$$

$$+\begin{cases} \rho_3 \left\langle \theta_1, \theta_2 \right\rangle_{L^2(0,1)} & \text{in case (1.1)}, \\ \beta \left\langle \theta_{1x}, \theta_{2x} \right\rangle_{L^2(0,1)} + \rho_3 \left\langle \tilde{\theta}_1, \tilde{\theta}_2 \right\rangle_{L^2(0,1)} & \text{in case (1.4)}, \end{cases}$$

where

$$\Phi_j = \begin{cases} (\varphi_j,\, \tilde{\varphi}_j,\, \psi_j,\, \tilde{\psi}_j,\, w_j,\, \tilde{w}_j,\, \theta_j)^T & \text{in case (1.1)}, \\ (\varphi_j,\, \tilde{\varphi}_j,\, \psi_j,\, \tilde{\psi}_j,\, w_j,\, \tilde{w}_j,\, \theta_j,\, \tilde{\theta}_j)^T & \text{in case (1.4)}, \end{cases} \qquad j = 1, 2.$$

From the definition of $H_*^1(0,1)$ and $\widetilde{H}_*^1(0,1)$, we notice that, if $(\varphi, \psi, w) \in H_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1)$ satisfying

$$k \left\| (\varphi_x + \psi + l\, w) \right\|_{L^2(0,1)}^2 + b \left\| \psi_x \right\|_{L^2(0,1)}^2 + k_0 \left\| (w_x - l\varphi) \right\|_{L^2(0,1)}^2 = 0,$$

then $\psi = 0$,

$$\varphi(x) = -c \sin(lx) \quad \text{and} \quad w(x) = c \cos(lx),$$

where c is a constant such that $c = 0$ or $l = \frac{\pi}{2} + m\pi$, for some $m \in \mathbb{N}$. So, if

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}, \tag{2.1}$$

then $\varphi = w = 0$. Here and after we assume that (2.1) is satisfied. Thus, $\mathcal{H}$ is a Hilbert space.

We introduce also the vectors

$$\Phi = \begin{cases} \left(\varphi,\, \tilde{\varphi},\, \psi,\, \tilde{\psi},\, w,\, \tilde{w},\, \theta\right)^T & \text{in case (1.1)}, \\ \left(\varphi,\, \tilde{\varphi},\, \psi,\, \tilde{\psi},\, w,\, \tilde{w},\, \theta,\, \tilde{\theta}\right)^T & \text{in case (1.4)} \end{cases}$$

and

$$\Phi_0 = \begin{cases} (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1,\, w_0,\, w_1,\, \theta_0)^T & \text{in case (1.1)}, \\ (\varphi_0,\, \varphi_1,\, \psi_0,\, \psi_1,\, w_0,\, w_1,\, \theta_0,\, \theta_1)^T & \text{in case (1.4)}, \end{cases}$$

where $\tilde{\varphi} = \varphi_t$, $\tilde{\psi} = \psi_t$, $\tilde{w} = w_t$ and $\tilde{\theta} = \theta_t$. Systems (1.1) – (1.3) and (1.2) – (1.5) can be written as a first order system given by

$$\begin{cases} \Phi_t = \mathcal{A}\Phi & \text{in } (0, \infty), \\ \Phi(t = 0) = \Phi_0, \end{cases} \tag{2.2}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\Phi = \begin{pmatrix} \tilde{\varphi} \\ \dfrac{k}{\rho_1}(\varphi_x + \psi + l\, w)_x + \dfrac{lk_0}{\rho_1}(w_x - l\varphi) - \dfrac{\delta}{\rho_1}\theta_x \\ \tilde{\psi} \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi + l\, w) \\ \tilde{w} \\ \dfrac{k_0}{\rho_1}(w_x - l\varphi)_x - \dfrac{lk}{\rho_1}(\varphi_x + \psi + l\, w) \\ \dfrac{\beta}{\rho_3}\theta_{xx} - \dfrac{\delta}{\rho_3}\tilde{\varphi}_x \end{pmatrix}$$

in case (1.1), and

$$\mathcal{A}\Phi=\begin{pmatrix} \tilde{\varphi} \\ \dfrac{k}{\rho_1}\left(\varphi_x+\psi+l\,w\right)_x+\dfrac{lk_0}{\rho_1}\left(w_x-l\varphi\right)-\dfrac{\delta}{\rho_1}\tilde{\theta}_x \\ \tilde{\psi} \\ \dfrac{b}{\rho_2}\psi_{xx}-\dfrac{k}{\rho_2}\left(\varphi_x+\psi+l\,w\right) \\ \tilde{w} \\ \dfrac{k_0}{\rho_1}\left(w_x-l\varphi\right)_x-\dfrac{lk}{\rho_1}\left(\varphi_x+\psi+l\,w\right) \\ \tilde{\theta} \\ \dfrac{\beta}{\rho_3}\theta_{xx}+\dfrac{\gamma}{\rho_3}\tilde{\theta}_{xx}-\dfrac{\delta}{\rho_3}\tilde{\varphi}_x \end{pmatrix}$$

in case (1.4). The domain of $\mathcal{A}$ is given by

$$D\left(\mathcal{A}\right)=\left\{\begin{array}{c} \Phi\in\mathcal{H}\mid\ \varphi\in\ H_*^2\left(0,1\right);\ \psi,\ w,\ \theta\in\ \widetilde{H}_*^2\left(0,1\right);\ \tilde{\varphi}\in H_*^1\left(0,1\right); \\ \tilde{\psi},\ \tilde{w}\in\ \widetilde{H}_*^1\left(0,1\right);\ \varphi_x\left(1\right)=\psi_x\left(0\right)=w_x\left(0\right)=\theta_x\left(0\right)=0 \end{array}\right\}$$

in case (1.1), and

$$D\left(\mathcal{A}\right)=\left\{\begin{array}{c} \Phi\in\mathcal{H}\mid\ \varphi\in\ H_*^2\left(0,1\right);\ \psi,\ w,\ \beta\theta+\gamma\tilde{\theta}\in\ \widetilde{H}_*^2\left(0,1\right);\ \tilde{\varphi}\in H_*^1\left(0,1\right); \\ \tilde{\psi},\ \tilde{w},\ \tilde{\theta}\in\ \widetilde{H}_*^1\left(0,1\right);\ \varphi_x\left(1\right)=\psi_x\left(0\right)=w_x\left(0\right)=\theta_x\left(0\right)=0 \end{array}\right\}$$

in case (1.4).

Now, we prove that the operator $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$. A direct calculation gives

$$\left\langle\mathcal{A}\Phi,\Phi\right\rangle_{\mathcal{H}}=\begin{cases} -\beta\left\|\theta_x\right\|^2_{L^2(0,1)} & \text{in case (1.1)}, \\ -\gamma\left\|\tilde{\theta}_x\right\|^2_{L^2(0,1)} & \text{in case (1.4)}. \end{cases}\tag{2.3}$$

Hence, $\mathcal{A}$ is dissipative in $\mathcal{H}$. On the other hand, we show that $0\in\rho\left(\mathcal{A}\right)$; that is, for any $F\in\mathcal{H}$, there exists $Z\in D\left(\mathcal{A}\right)$ satisfying

$$\mathcal{A}Z=F.\tag{2.4}$$

2.1. **Case of system** (1.1) – (1.3)**.** Let $F=\left(f_1,\cdots,f_7\right)^T$ and $Z=\left(z_1,\cdots,z_7\right)^T$. The first, third and fifth equations in (2.4) are equivalent to

$$z_2=f_1,\quad z_4=f_3\quad\text{and}\quad z_6=f_5,\tag{2.5}$$

and then, because $F\in\mathcal{H}$,

$$z_2\in H_*^1\left(0,1\right)\quad\text{and}\quad z_4,\ z_6\in\ \widetilde{H}_*^1\left(0,1\right).\tag{2.6}$$

Second, substitute z_2 into the last equation in (2.4), we conclude that the last equation in (2.4) is reduced to

$$z_{7xx}=\frac{\delta}{\beta}f_{1x}+\frac{\rho_3}{\beta}f_7.\tag{2.7}$$

By a direct integration, we see that (2.7) has a unique solution z_7 satisfying

$$z_7\in\ \widetilde{H}_*^2\left(0,1\right)\quad\text{and}\quad z_{7x}(0)=0,\tag{2.8}$$

this solution is given by

$$z_7(x)=\frac{1}{\beta}\int_1^x\int_0^y\left[\delta f_{1x}(\tau)+\rho_3 f_7(\tau)\right]\,d\tau\,dy.$$

Finally, the second, fourth and sixth equations in (2.4) become

$$\begin{cases} k\left(z_{1x}+z_3+l\,z_5\right)_x+lk_0\left(z_{5x}-lz_1\right)=\delta z_{7x}+\rho_1 f_2, \\ bz_{3xx}-k\left(z_{1x}+z_3+l\,z_5\right)=\rho_2 f_4, \\ k_0\left(z_{5x}-lz_1\right)_x-lk\left(z_{1x}+z_3+l\,z_5\right)=\rho_1 f_6. \end{cases} \tag{2.9}$$

To prove that (2.9) admits a solution (z_1, z_3, z_5) satisfying

$$z_1\in H_*^2\left(0,1\right),\quad z_3,\, z_5\in \widetilde{H}_*^2\left(0,1\right)\quad \text{and}\quad z_{1x}(1)=z_{3x}(0)=z_{5x}(0)=0, \tag{2.10}$$

we put

$$\mathcal{H}_0=H_*^1\left(0,1\right)\times \widetilde{H}_*^1\left(0,1\right)\times \widetilde{H}_*^1\left(0,1\right)$$

and we define the bilinear form on $\mathcal{H}_0\times\mathcal{H}_0$

$$\begin{aligned} a\left(\left(v_1,\, v_2,\, v_3\right),\left(w_1,\, w_2,\, w_3\right)\right)&=k\left\langle v_{1x}+v_2+lv_3, w_{1x}+w_2+lw_3\right\rangle_{L^2(0,1)} \\ &\quad +b\left\langle v_{2x},w_{2x}\right\rangle_{L^2(0,1)}+k_0\left\langle v_{3x}-lv_1,w_{3x}-lw_1\right\rangle_{L^2(0,1)} \end{aligned}$$

and the linear form on $\mathcal{H}_0$

$$l_1\left(v_1,\, v_2,\, v_3\right)=\left\langle \delta z_{7x}+\rho_1 f_2, v_1\right\rangle_{L^2(0,1)}+\left\langle \rho_2 f_4, v_2\right\rangle_{L^2(0,1)}+\left\langle \rho_1 f_6, v_3\right\rangle_{L^2(0,1)}.$$

Thus, the variational formulation of (2.9) is given by

$$a\left(\left(z_1,\, z_3,\, z_5\right),\left(w_1,\, w_2,\, w_3\right)\right)=l_1\left(w_1,\, w_2,\, w_3\right),\,\forall\left(w_1,\, w_2,\, w_3\right)^T\in\mathcal{H}_0. \tag{2.11}$$

From the Lax-Milgram theorem, it follows that (2.11) has a unique solution

$$\left(z_1,\, z_3,\, z_5\right)\in\mathcal{H}_0.$$

Therefore, using classical elliptic regularity arguments, we conclude that $(z_1,\, z_3,\, z_5)$ solves (2.9) and satisfies the regularity and boundary conditions (2.10). This proves that (2.4) has a unique solution $Z\in D\left(\mathcal{A}\right)$. By the resolvent identity, we have $\lambda I-\mathcal{A}$ is surjective, for any $\lambda>0$ (see [20]), where I denotes the identity operator. Consequently, the Lumer-Phillips theorem implies that $\mathcal{A}$ is the infinitesimal generator of a linear C_0 semigroup of contractions on $\mathcal{H}$.

2.2. **Case of system** $(1.2)-(1.5)$**.** Let $F=(f_1,\cdots,f_8)^T$ and $Z=(z_1,\cdots,z_8)^T$. The first, third, fifth and seventh equations in (2.4) are equivalent to

$$z_2=f_1,\quad z_4=f_3,\quad z_6=f_5\quad \text{and}\quad z_8=f_7, \tag{2.12}$$

and then, because $F\in\mathcal{H}$,

$$z_2\in H_*^1\left(0,1\right)\quad \text{and}\quad z_4,\, z_6,\, z_8\in \widetilde{H}_*^1\left(0,1\right). \tag{2.13}$$

Second, substitute z_2 and z_8 into the last equation in (2.4), we conclude that the last equation in (2.4) is reduced to

$$\left(\beta z_7+\gamma f_7\right)_{xx}=\delta f_{1x}+\rho_3 f_8. \tag{2.14}$$

Because $\delta f_{1x}+\rho_3 f_8\in L^2(0,1)$ and $f_7\in \widetilde{H}_*^1\left(0,1\right)$, then, by classical arguments, we see that (2.14) has a unique solution z_7 satisfying

$$\beta z_7+\gamma f_7\in \widetilde{H}_*^2\left(0,1\right),\quad z_7\in \widetilde{H}_*^1\left(0,1\right)\quad \text{and}\quad z_{7x}(0)=0. \tag{2.15}$$

Finally, the second, fourth and sixth equations in (2.4) become

$$\begin{cases} k\left(z_{1x}+z_3+l\,z_5\right)_x+lk_0\left(z_{5x}-lz_1\right)=\delta f_{7x}+\rho_1 f_2, \\ bz_{3xx}-k\left(z_{1x}+z_3+l\,z_5\right)=\rho_2 f_4, \\ k_0\left(z_{5x}-lz_1\right)_x-lk\left(z_{1x}+z_3+l\,z_5\right)=\rho_1 f_6. \end{cases} \tag{2.16}$$

To prove that (2.16) admits a solution (z_1, z_3, z_5) satisfying (2.10), we follow the same arguments as in the previous case by considering the variational formulation of (2.16) given by

$$a\left(\left(z_1,\, z_3,\, z_5\right),\left(w_1,\, w_2,\, w_3\right)\right)=l_2\left(w_1,\, w_2,\, w_3\right),\,\forall\left(w_1,\, w_2,\, w_3\right)^T\in\mathcal{H}_0,$$

where

$$l_2(v_1, v_2, v_3) = \langle \delta f_{7x} + \rho_1 f_2, v_1\rangle_{L^2(0,1)} + \langle \rho_2 f_4, v_2\rangle_{L^2(0,1)} + \langle \rho_1 f_6, v_3\rangle_{L^2(0,1)} .$$

Consequently, the following well-posedness results for (2.2) hold (see [22]):

Theorem 2.1. *Assume that* (2.1) *holds. Then, for any* $m \in \mathbb{N}$ *and* $\Phi_0 \in D(\mathcal{A}^m)$*, system* (2.2) *admits a unique solution*

$$\Phi \in \cap_{j=0}^{m} C^{m-j}\left(\mathbb{R}_+; D\left(\mathcal{A}^j\right)\right). \tag{2.17}$$

In the next two sections, we will show the non-exponential and plynomial stability of (2.2), where the proof is based on the following theorems:

Theorem 2.2. *(*[14] *and* [23]*) A C_0 semigroup of contractions on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{\lambda\in\mathbb{R}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.18}$$

Theorem 2.3. *(*[18]*) If a bounded C_0 semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ satisfies, for some $j \in \mathbb{N}^*$,*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\lambda|\geq 1} \frac{1}{\lambda^j} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.19}$$

Then, for any $m \in \mathbb{N}^*$, there exists a positive constant c_m such that

$$\left\|e^{t\mathcal{A}} z_0\right\|_{\mathcal{H}} \leq c_m \|z_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t}\right)^{\frac{m}{j}} \ln t, \quad \forall z_0 \in D(\mathcal{A}^m), \ \forall t > 0. \tag{2.20}$$

3. Lack of exponential stability

Our objective here is to show that the semigroup associated with (2.2) is not exponentailly stable.

Theorem 3.1. *We assume that* (2.1) *holds. Then, the semigroup associated with* (2.2) *is not exponentially stable.*

Proof. We use Theorem 2.2 by proving that the second condition in (2.18) is not satisfied; that is we prove that there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \left\|(i\lambda_n I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \infty,$$

which is equivalent to prove that there exists a sequence $(F_n)_n \subset \mathcal{H}$ satisfying

$$\|F_n\|_{\mathcal{H}} \leq 1, \quad \forall n \in \mathbb{N} \tag{3.1}$$

and

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})^{-1} F_n\|_{\mathcal{H}} = \infty. \tag{3.2}$$

For this purpose, let

$$\Phi_n = (i\lambda_n I - \mathcal{A})^{-1} F_n, \quad \forall n \in \mathbb{N}.$$

Then, we have to prove that (3.1) holds such that

$$\lim_{n\to\infty} \|\Phi_n\|_{\mathcal{H}} = \infty \quad \text{and} \quad i\lambda_n \Phi_n - \mathcal{A}\Phi_n = F_n, \ \forall n \in \mathbb{N}. \tag{3.3}$$

Taking

$$\Phi_n = \begin{cases} \left(\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n, \theta_n\right)^T & \text{in case (1.1)}, \\ \left(\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n, \theta_n, \tilde{\theta}_n\right)^T & \text{in case (1.4)} \end{cases}$$

and

$$F_n = \begin{cases} (f_{1n}, \cdots, f_{7n})^T & \text{in case (1.1)}, \\ (f_{1n}, \cdots, f_{8n})^T & \text{in case (1.4)}. \end{cases}$$

Therefore, from the second equality in (3.3), we have the following systems:

$$
\begin{cases}
i\lambda_n\varphi_n - \tilde{\varphi}_n = f_{1n}, \\
i\rho_1\lambda_n\tilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) + \delta\theta_{nx} = \rho_1 f_{2n}, \\
i\lambda_n\psi_n - \tilde{\psi}_n \ = f_{3n}, \\
i\rho_2\lambda_n\tilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\
i\lambda_n w_n - \tilde{w}_n = f_{5n}, \\
i\rho_1\lambda_n\tilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_1 f_{6n}, \\
i\rho_3\lambda_n\theta_n - \beta\theta_{nxx} + \delta\tilde{\varphi}_{nx} = \rho_3 f_{7n}
\end{cases} \tag{3.4}
$$

in case (1.1), and

$$
\begin{cases}
i\lambda_n\varphi_n - \tilde{\varphi}_n = f_{1n}, \\
i\rho_1\lambda_n\tilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) + \delta\tilde{\theta}_{nx} = \rho_1 f_{2n}, \\
i\lambda_n\psi_n - \tilde{\psi}_n \ = f_{3n}, \\
i\rho_2\lambda_n\tilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\
i\lambda_n w_n - \tilde{w}_n = f_{5n}, \\
i\rho_1\lambda_n\tilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_1 f_{6n}, \\
i\lambda_n\theta_n - \tilde{\theta}_n = f_{7n}, \\
i\rho_3\lambda_n\tilde{\theta}_n - \beta\theta_{nxx} - \gamma\tilde{\theta}_{nxx} + \delta\tilde{\varphi}_{nx} = \rho_3 f_{8n}
\end{cases} \tag{3.5}
$$

in case (1.4). Choosing

$$
\begin{cases}
f_{1n} = f_{3n} = f_{5n} = 0 & \text{in case}\,(1.1), \\
f_{1n} = f_{3n} = f_{5n} = f_{7n} = 0 & \text{in case}\,(1.4).
\end{cases} \tag{3.6}
$$

Thus, systems (3.4) and (3.5) become, respectively,

$$
\begin{cases}
\tilde{\varphi}_n = i\lambda_n\varphi_n, \quad \tilde{\psi}_n = i\lambda_n\psi_n, \quad \tilde{w}_n = i\lambda_n w_n, \\
-\rho_1\lambda_n^2\varphi_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) + \delta\theta_{nx} = \rho_1 f_{2n}, \\
-\rho_2\lambda_n^2\psi_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\
-\rho_1\lambda_n^2 w_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_1 f_{6n}, \\
i\rho_3\lambda_n\theta_n - \beta\theta_{nxx} + i\delta\lambda_n\varphi_{nx} = \rho_3 f_{7n}
\end{cases} \tag{3.7}
$$

and

$$
\begin{cases}
\tilde{\varphi}_n = i\lambda_n\varphi_n, \quad \tilde{\psi}_n = i\lambda_n\psi_n, \quad \tilde{w}_n = i\lambda_n w_n, \quad \tilde{\theta}_n = i\lambda_n\theta_n, \\
-\rho_1\lambda_n^2\varphi_n - k\left(\varphi_{nx} + \psi_n + l\,w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) + i\delta\lambda_n\theta_{nx} = \rho_1 f_{2n}, \\
-\rho_2\lambda_n^2\psi_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_2 f_{4n}, \\
-\rho_1\lambda_n^2 w_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\,w_n\right) = \rho_1 f_{6n}, \\
-i\rho_3\lambda_n^2\theta_n - \beta\theta_{nxx} - i\gamma\lambda_n\theta_{nxx} + i\delta\lambda_n\varphi_{nx} = \rho_3 f_{8n}.
\end{cases} \tag{3.8}
$$

To simplify the calculations, we put $N = \frac{(2n+1)\pi}{2}$ and consider few cases. Some of the next computations were given in [1] and will be addapted here to our problems.

Case 1: $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$. We choose

$$
f_{4n}(x) = -\frac{lk_0}{\rho_2} D\cos\,(Nx)\,, \;\; f_{6n}(x) = -\frac{l^2 k_0}{\rho_1} D\cos\,(Nx) \tag{3.9}
$$

and

$$\begin{cases} f_{2n} = f_{7n} = 0 & \text{in case (1.1)}, \\ f_{2n} = f_{8n} = 0 & \text{in case (1.4)}, \end{cases} \tag{3.10}$$

where $D \in \mathbb{R}$. We will look for $\Phi_n \in D(\mathcal{A})$ such that

$$\begin{cases} \varphi_n = \tilde{\varphi}_n = \theta_n = \tilde{\theta}_n = 0, \ \psi_n(x) = B\cos(Nx), \ \tilde{\psi}_n(x) = iB\lambda_n \cos(Nx), \\ w_n(x) = D\cos(Nx) \quad \text{and} \quad \tilde{w}_n(x) = iD\lambda_n \cos(Nx), \end{cases}$$

where $B \in \mathbb{R}$. Systems (3.7) and (3.8) are satisfied if and only if

$$\begin{cases} kB + l(k+k_0)D = 0, \\ \left[-\lambda_n^2 + \dfrac{b}{\rho_2}N^2 + \dfrac{k}{\rho_2}\right]B + \dfrac{lk}{\rho_2}D = -\dfrac{lk_0}{\rho_2}D, \\ \dfrac{lk}{\rho_1}B + \left[-\lambda_n^2 + \dfrac{k_0}{\rho_1}N^2 + \dfrac{l^2k}{\rho_1}\right]D = -\dfrac{l^2k_0}{\rho_1}D. \end{cases} \tag{3.11}$$

Taking

$$\lambda_n = N\sqrt{\frac{k_0}{\rho_1}}. \tag{3.12}$$

Because $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$, we get

$$-\lambda_n^2 + \frac{b}{\rho_2}N^2 = -\lambda_n^2 + \frac{k_0}{\rho_1}N^2 = 0,$$

and therefore, the system (3.11) will be reduced to

$$kB + l(k+k_0)D = 0,$$

which is equivalent to

$$B = -l\left(1 + \frac{k_0}{k}\right)D. \tag{3.13}$$

Choosing

$$D = \frac{\rho_1\rho_2}{lk_0\sqrt{\rho_1^2 + l^2\rho_2^2}}$$

and using (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned} \|F_n\|_{\mathcal{H}}^2 = \|f_{4n}\|_{L^2(0,1)}^2 + \|f_{6n}\|_{L^2(0,1)}^2 &= \left(\frac{lk_0}{\rho_2}\right)^2\left[1 + \left(\frac{l\rho_2}{\rho_1}\right)^2\right]D^2\int_0^1 \cos^2(Nx)\,dx \\ &\le \left(\frac{lk_0}{\rho_2}\right)^2\left[1 + \left(\frac{l\rho_2}{\rho_1}\right)^2\right]D^2 = 1, \end{aligned}$$

so, (3.1) is satisfied. On the other hand, we have

$$\begin{aligned} \|\Phi_n\|_{\mathcal{H}}^2 &\ge k_0\|w_{nx} - l\varphi_n\|_{L^2(0,1)}^2 = k_0\|w_{nx}\|_{L^2(0,1)}^2 \\ &\ge \frac{k_0}{2}D^2N^2\int_0^1 [1 - \cos(2Nx)]\,dx = \frac{k_0}{2}D^2N^2, \end{aligned}$$

hence, the limit in (3.3) holds.

Case 2: $\dfrac{b}{\rho_2} \neq \dfrac{k_0}{\rho_1}$ and $k \neq k_0$. We consider $(3.6)_1$ and choose

$$f_{2n} = f_{4n} = f_{7n} = 0, \quad f_{6n}(x) = \cos(Nx), \tag{3.14}$$

and

$$\text{(3.15)} \qquad \begin{cases} \varphi_n(x) = \alpha_1 \sin\,(Nx)\,, & \tilde{\varphi}_n(x) = i\alpha_1\lambda_n \sin\,(Nx)\,, \\ \psi_n(x) = \alpha_2 \cos\,(Nx)\,, & \tilde{\psi}_n(x) = i\alpha_2\lambda_n \cos\,(Nx)\,, \\ w_n(x) = \alpha_3 \cos\,(Nx)\,, & \tilde{w}_n(x) = i\alpha_3\lambda_n \cos\,(Nx)\,, \\ \theta_n(x) = \alpha_4 \cos\,(Nx) \end{cases}$$

in case (1.1), and $(3.6)_2$, (3.14), (3.15),

$$\text{(3.16)} \qquad f_{8n} = 0 \quad \text{and} \quad \tilde{\theta}_n(x) = i\alpha_4\lambda_n \cos\,(Nx)$$

in case (1.4), where $\alpha_1, \cdots, \alpha_4$ are constants depending on N. For λ_n, we consider the choice

$$\text{(3.17)} \qquad \lambda_n = \sqrt{\frac{k_0}{\rho_1}N^2 + \frac{l^2 k}{\rho_1}}.$$

According to these choices, we see that $\Phi_n \in D(\mathcal{A})$, $F_n \in \mathcal{H}$ and

$$\text{(3.18)} \qquad \|F_n\|_{\mathcal{H}}^2 = \|f_{6n}\|_{L^2(0,1)}^2 = \int_0^1 \cos^2\,(Nx)\; dx \le 1,$$

which gives (3.1). On the other hand, thanks to the above choices, (3.7) and (3.8) are satisfied if and only if

$$\alpha_4 = \mu_n N^2 \alpha_1$$

and

$$\text{(3.19)} \qquad \begin{cases} \left[(k-\mu_n)\,N^2 - \rho_1\lambda_n^2 + l^2 k_0\right]\alpha_1 + kN\alpha_2 + l\,(k+k_0)\,N\alpha_3 = 0, \\ kN\alpha_1 + \left(bN^2 - \rho_2\lambda_n^2 + k\right)\alpha_2 + kl\alpha_3 = 0, \\ l\,(k+k_0)\,N\alpha_1 + lk\alpha_2 + \left(k_0N^2 - \rho_1\lambda_n^2 + l^2k\right)\alpha_3 = \rho_1, \end{cases}$$

where

$$\text{(3.20)} \qquad \mu_n = \begin{cases} \frac{-i\delta^2\lambda_n}{\beta N^2 + i\rho_3\lambda_n} & \text{in case}\,(1.1), \\ \frac{\delta^2\lambda_n^2}{i\gamma\lambda_n N^2 + \beta N^2 - \rho_3\lambda_n^2} & \text{in case}\,(1.4). \end{cases}$$

From the choice (3.17), we remark that the last equation in (3.19) is equivalent to

$$\text{(3.21)} \qquad \alpha_2 = -\frac{k+k_0}{k}N\alpha_1 + \frac{\rho_1}{lk},$$

so, substituting in the first two equations in (3.19), we entail

$$\text{(3.22)} \qquad \alpha_3 = a_1 N\alpha_1 + a_2$$

and

$$\text{(3.23)} \qquad \alpha_1 = \frac{\left[l\,(k+k_0)\,a_2 + \frac{\rho_1}{l}\right]N}{\left[2k_0 + \mu_n - l\,(k+k_0)\,a_1\right]N^2 + l^2\,(k-k_0)},$$

where

$$\begin{cases} a_1 = \frac{k+k_0}{lk^2}\left(b - \frac{\rho_2 k_0}{\rho_1}\right)N^2 + \frac{k_0}{lk} - \frac{l\rho_2(k+k_0)}{\rho_1 k}, \\ a_2 = \frac{\rho_1}{(lk)^2}\left[\left(\frac{\rho_2 k_0}{\rho_1} - b\right)N^2 + \frac{l^2\rho_2 k}{\rho_1} - k\right]. \end{cases}$$

To simplify the computations, we put

$$\begin{cases} a_3 = \frac{\rho_1\,(k+k_0)}{lk^2}\left(\frac{\rho_2 k_0}{\rho_1} - b\right), \quad a_4 = \frac{(k+k_0)^2}{k^2}\left(\frac{\rho_2 k_0}{\rho_1} - b\right), \\ a_5 = \frac{l\rho_2(k+k_0)}{k} - \frac{k_0\rho_1}{lk}, \quad a_6 = \frac{l^2\rho_2\,(k+k_0)^2}{\rho_1 k} + \frac{k_0\,(k-k_0)}{k} \end{cases}$$

and

$$\begin{cases} d_0 = \frac{k+k_0}{lk^2}\left(b - \dfrac{\rho_2 k_0}{\rho_1}\right), & d_1 = \frac{\rho_1}{(lk)^2}\left(\dfrac{\rho_2 k_0}{\rho_1} - b\right), \\ d_2 = \dfrac{k_0}{lk} - \dfrac{l\rho_2(k+k_0)}{\rho_1 k}, & d_3 = \frac{\rho_1}{l^2 k}\left(\dfrac{l^2\rho_2}{\rho_1} - 1\right). \end{cases}$$

It follows that

$$N\alpha_1 = \frac{a_3 N^4 + a_5 N^2}{a_4 N^4 + (\mu_n + a_6) N^2 + l^2 (k - k_0)}$$

and (notice that $d_0 a_3 + d_1 a_4 = 0$)

$$\alpha_3 = \frac{\left(d_0 N^2 + d_2\right)\left(a_3 N^4 + a_5 N^2\right)}{a_4 N^4 + (\mu_n + a_6) N^2 + l^2 (k - k_0)} + d_1 N^2 + d_3 \tag{3.24}$$

$$= \frac{(d_0 a_5 + d_2 a_3 + d_3 a_4 + d_1 a_6 + d_1 \mu_n) N^4 + \left(d_2 a_5 + d_3 a_6 + l^2 (k - k_0) d_1 + d_3 \mu_n\right) N^2 + l^2 (k - k_0) d_3}{a_4 N^4 + (\mu_n + a_6) N^2 + l^2 (k - k_0)},$$

Because $\dfrac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$ and $k \neq k_0$, it appears that $a_4 \neq 0$ and

$$d_0 a_5 + d_2 a_3 + d_3 a_4 + d_1 a_6 = \frac{\rho_1}{(lk)^2}\left(\frac{\rho_2 k_0}{\rho_1} - b\right)(k_0 - k) \neq 0. \tag{3.25}$$

On the other hand, we have

$$\lim_{n\to\infty} \mu_n = 0, \tag{3.26}$$

then, we deduce from (3.24), (3.25) and (3.26) that

$$\lim_{n\to\infty} \alpha_3 = \frac{d_0 a_5 + d_2 a_3 + d_3 a_4 + d_1 a_6}{a_4} \neq 0, \tag{3.27}$$

hence,

$$\lim_{n\to\infty} |\alpha_3| \lambda_n = \infty. \tag{3.28}$$

Now, we notice that

$$\|\Phi_n\|_{\mathcal{H}}^2 \geq \rho_1 \|\tilde{w}_n\|_{L^2(0,1)}^2 = \rho_1 (|\alpha_3|\lambda_n)^2 \int_0^1 \cos^2 (Nx)\, dx$$

$$\geq \frac{1}{2}\rho_1 (|\alpha_3|\lambda_n)^2 \int_0^1 [1 + \cos(2Nx)]\, dx = \frac{1}{2}\rho_1 (|\alpha_3|\lambda_n)^2,$$

thus, by (3.28) we infer that the limit in (3.3).

Case 3: $\dfrac{b}{\rho_2} \neq \dfrac{k_0}{\rho_1}$ and $k = k_0$. We consider the choices $(3.6)_1$,

$$\lambda_n = \sqrt{\frac{b}{\rho_2} N^2 + \frac{k}{2\rho_2}}, \tag{3.29}$$

$$f_{2n} = f_{7n} = 0, \quad f_{4n}(x) = \alpha_2 C_n \cos(Nx), \quad f_{6n}(x) = \alpha_2 D_n \cos(Nx), \tag{3.30}$$

and (3.15) with

$$\alpha_1 = \left(\frac{\rho_1 D_n}{2lk} - \frac{1}{2}\right)\frac{\alpha_2}{N}, \quad \alpha_3 = 0 \quad \text{and} \quad \alpha_4 = \mu_n N^2 \alpha_1 \tag{3.31}$$

in case (1.1), and $(3.6)_2$, (3.15), (3.16), (3.29), (3.30) and (3.31) in case (1.4), where

$$C_n = \frac{\rho_1}{2l\rho_2} D_n, \quad D_n = \frac{2lk}{\rho_1}\left(\frac{1}{2} - \frac{k}{k + \frac{l^2 k}{N^2} - \mu_n - \frac{\rho_1 \lambda_n^2}{N^2}}\right)$$

and μ_n is defined in (3.20). According to (3.20) and (3.29), we remark that (3.26) holds, and moreover

$$\lim_{n\to\infty} D_n = \frac{2lk}{\rho_1}\left(\frac{1}{2} - \frac{k}{k-\frac{\rho_1 b}{\rho_2}}\right) \quad \text{and} \quad \lim_{n\to\infty} C_n = \frac{k}{\rho_2}\left(\frac{1}{2} - \frac{k}{k-\frac{\rho_1 b}{\rho_2}}\right)$$

(these limits exist since $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$ and $k = k_0$), so, the sequence $\left(|C_n|^2 + |D_n|^2\right)_n$ is bounded. Then, we choose

$$\alpha_2 = \frac{1}{\sqrt{sup_{n\in\mathbb{N}}\left(|C_n|^2 + |D_n|^2\right)}}. \tag{3.32}$$

According to these choices, it is clear that $\Phi_n \in D(\mathcal{A})$, $F_n \in \mathcal{H}$ and

$$\|F_n\|_{\mathcal{H}}^2 = \|f_{4n}\|_{L^2(0,1)}^2 + \|f_{6n}\|_{L^2(0,1)}^2 = \left(|C_n|^2 + |D_n|^2\right)\alpha_2^2 \int_0^1 \cos^2(Nx)\,dx$$
$$\leq \left(|C_n|^2 + |D_n|^2\right)\alpha_2^2 \leq 1,$$

hence, (3.1) holds. On the other hand, because $k = k_0$ and $\alpha_3 = 0$, (3.7) and (3.8) are satisfied if and only if

$$\begin{cases} \left[(k-\mu_n)N^2 - \rho_1\lambda_n^2 + l^2 k\right]\alpha_1 + kN\alpha_2 = 0, \\ kN\alpha_1 + \left(bN^2 - \rho_2\lambda_n^2 + k\right)\alpha_2 = \rho_2\alpha_2 C_n, \\ 2lkN\alpha_1 + lk\alpha_2 = \rho_1\alpha_2 D_n. \end{cases} \tag{3.33}$$

The first equation in (3.33) is satisfied thanks to the definition of α_1 and D_n, the second equation in (3.33) holds according to the definition of λ_n, α_1 and C_n, and the last equation in (3.33) is valid from the definition of α_1.

Now, we have

$$\|\Phi_n\|_{\mathcal{H}}^2 \geq \rho_2 \left\|\tilde{\psi}_n\right\|_{L^2(0,1)}^2 = \rho_2(\alpha_2\lambda_n)^2 \int_0^1 \cos^2(Nx)\,dx$$
$$\geq \frac{1}{2}\rho_2(\alpha_2\lambda_n)^2 \int_0^1 [1+\cos(2Nx)]\,dx = \frac{1}{2}\rho_2(\alpha_2\lambda_n)^2,$$

consequently, the limit in (3.3) holds.

Finally, there exist sequences $(F_n)_n \subset \mathcal{H}$, $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying (3.1) and (3.3). Hence, Theorem 2.2 implies that system (2.2) is not exponentially stable. □

4. Polynomial stability

In this section, we use Theorem 2.3 to prove that the semigroup associated to (2.2) is polynomially stable. Our main result is stated as follow:

Theorem 4.1. *Assume that* (2.1) *holds and*

$$l^2 \neq \frac{k_0\rho_2 - b\rho_1}{k_0\rho_2}\left(\frac{\pi}{2} + m\pi\right)^2 - \frac{k\rho_1}{\rho_2(k+k_0)}, \quad \forall m \in \mathbb{Z}. \tag{4.1}$$

Then, for any $m \in \mathbb{N}^*$, there exists a constant $c_m > 0$ such that

$$\forall \Phi_0 \in D(\mathcal{A}^m),\ \forall t > 0,\ \left\|e^{t\mathcal{A}}\Phi_0\right\|_{\mathcal{H}} \leq c_m \|\Phi_0\|_{D(\mathcal{A}^m)} \left(\frac{\ln t}{t}\right)^{\frac{m}{4}} \ln t. \tag{4.2}$$

Proof. We start by proving that

$$i\,\mathbb{R} \subset \rho(\mathcal{A}) \tag{4.3}$$

is equivalent to (4.1). In section 2, we have proved that $0 \in \rho(\mathcal{A})$. So, let $\lambda \in \mathbb{R}^*$. We prove that $i\lambda$ is not an eigenvalue of $\mathcal{A}$ by proving that the unique solution

$$\Phi = \begin{cases} \left(\varphi,\, \tilde{\varphi},\, \psi,\, \tilde{\psi},\, w,\, \tilde{w},\, \theta\right)^T & \text{in case}\,(1.1), \\ \left(\varphi,\, \tilde{\varphi},\, \psi,\, \tilde{\psi},\, w,\, \tilde{w},\, \theta,\, \tilde{\theta}\right)^T & \text{in case}\,(1.4) \end{cases}$$

in $D(\mathcal{A})$ of the equation

$$\mathcal{A}\,\Phi = i\,\lambda\,\Phi \tag{4.4}$$

is $\Phi = 0$. The equation (4.4) means that

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \\ \dfrac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \dfrac{lk_0}{\rho_1}\left(w_x - l\varphi\right) - \dfrac{\delta}{\rho_1}\theta_x = i\lambda\tilde{\varphi}, \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) = i\lambda\tilde{\psi}, \\ \dfrac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \dfrac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) = i\lambda\tilde{w}, \\ \dfrac{\beta}{\rho_3}\theta_{xx} - \dfrac{\delta}{\rho_3}\tilde{\varphi}_x = i\lambda\theta \end{cases} \tag{4.5}$$

in case (1.1), and

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \quad \tilde{\theta} = i\lambda\theta, \\ \dfrac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \dfrac{lk_0}{\rho_1}\left(w_x - l\varphi\right) - \dfrac{\delta}{\rho_1}\tilde{\theta}_x = i\lambda\tilde{\varphi}, \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) = i\lambda\tilde{\psi}, \\ \dfrac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \dfrac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) = i\lambda\tilde{w}, \\ \dfrac{\beta}{\rho_3}\theta_{xx} + \dfrac{\gamma}{\rho_3}\tilde{\theta}_{xx} - \dfrac{\delta}{\rho_3}\tilde{\varphi}_x = i\lambda\tilde{\theta} \end{cases} \tag{4.6}$$

in case (1.4). Using (2.3), we find

$$0 = Re\, i\lambda\, \|\Phi\|_{\mathcal{H}}^2 = Re\,\langle i\lambda\Phi, \Phi\rangle_{\mathcal{H}} = Re\,\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = \begin{cases} -\beta\,\|\theta_x\|_{L^2(0,1)}^2 & \text{in case}\,(1.1), \\ -\gamma\,\left\|\tilde{\theta}_x\right\|_{L^2(0,1)}^2 & \text{in case}\,(1.4). \end{cases}$$

Then,

$$\begin{cases} \theta_x = 0 & \text{in case}\,(1.1), \\ \tilde{\theta}_x = 0 & \text{in case}\,(1.4). \end{cases} \tag{4.7}$$

Taking into account that $\theta, \tilde{\theta} \in \widetilde{H_*^1}(0,1)$ (since $\Phi \in D(\mathcal{A})$) and the Poincaré's inequality and using (4.7) and the fourth equation in (4.6), we deduce that

$$\begin{cases} \theta = 0 & \text{in case}\,(1.1), \\ \theta = \tilde{\theta} = 0 & \text{in case}\,(1.4). \end{cases} \tag{4.8}$$

By using (4.8) and the last equation in (4.5) and (4.6), we arrive at

$$\tilde{\varphi}_x = 0. \tag{4.9}$$

Therefore, from the first equation in (4.5) and (4.6), we obtain

$$\varphi_x = 0. \tag{4.10}$$

As φ, $\tilde{\varphi} \in H_*^1(0,1)$ and according to the Poincaré's inequality, it follows that

$$\varphi = \tilde{\varphi} = 0. \tag{4.11}$$

Using (4.8) and (4.11), we see that (4.5) and (4.6) are reduced to

$$\begin{cases} \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \\ k\psi_x + l(k+k_0)w_x = 0, \\ b\psi_{xx} - k(\psi + l\,w) = -\rho_2\lambda^2\psi, \\ k_0 w_{xx} - lk(\psi + l\,w) = -\rho_1\lambda^2 w. \end{cases} \tag{4.12}$$

Taking into account that $(\psi, w) \in \left(\widetilde{H_*^1}(0,1)\right)^2$ and the Poincaré's inequality and using the third equation in (4.12), we entail

$$\psi = -l\left(1 + \frac{k_0}{k}\right)w. \tag{4.13}$$

Using the last two equations in (4.12), we infer that

$$lb\psi_{xx} - k_0 w_{xx} = -\rho_2 l\lambda^2\psi + \rho_1\lambda^2 w. \tag{4.14}$$

Then, combining with (4.13), we see that

$$w_{xx} + \alpha^2\lambda^2 w = 0,$$

where

$$\alpha = \sqrt{\frac{\rho_2 l^2(k+k_0) + k\rho_1}{bl^2(k+k_0) + kk_0}}. \tag{4.15}$$

This implies that, for c_1, $c_2 \in \mathbb{C}$,

$$w(x) = c_1\cos(\alpha\lambda x) + c_2\sin(\alpha\lambda x).$$

The boundary condition $w_x(0) = 0$ leads to $c_2 = 0$, and so, using (4.13),

$$\psi(x) = -l\left(1 + \frac{k_0}{k}\right)c_1\cos(\alpha\lambda x) \quad \text{and} \quad w(x) = c_1\cos(\alpha\lambda x). \tag{4.16}$$

Because $\psi(1) = w(1) = 0$, we have

$$c_1 = 0 \quad \text{or} \quad \exists m \in \mathbb{Z} : \alpha\lambda = \frac{\pi}{2} + m\pi.$$

If $c_1 = 0$, we get

$$\psi = w = 0. \tag{4.17}$$

Using (4.17) and the first two equations in (4.12), we arrive at

$$\tilde{\psi} = \tilde{w} = 0.$$

Consequently, $\Phi = 0$ and hence

$$i\lambda \in \rho(\mathcal{A}). \tag{4.18}$$

If $c_1 \neq 0$, we obtain

$$\exists m \in \mathbb{Z} : \alpha\lambda = \frac{\pi}{2} + m\pi. \tag{4.19}$$

Therefore, using (4.15) and (4.16), it appears that the last two equations in (4.12) are equivalent to

$$(k_0\rho_2 - b\rho_1)\lambda^2 = \frac{k_0}{k+k_0}\left[bl^2(k+k_0) + kk_0\right]. \tag{4.20}$$

Combining (4.15), (4.19) and (4.20), we entail

$$\exists m \in \mathbb{Z} : \left(\frac{\pi}{2} + m\pi\right)^2 = \frac{k_0\left[l^2\rho_2(k+k_0) + k\rho_1\right]}{(k+k_0)(k_0\rho_2 - b\rho_1)}. \tag{4.21}$$

If (4.1) holds, then (4.21) is impossible. Hence, $c_1 = 0$ and so (4.18) holds.

If (4.1) does not hold, then, for $\lambda = \frac{1}{\alpha}\left(\frac{\pi}{2} + m\pi\right)$ and for any $c_1 \in \mathbb{C}$,

$$\Phi(x) = \left(0, 0, -l\left(1+\frac{k_0}{k}\right)c_1 \cos(\alpha\lambda x), -il\left(1+\frac{k_0}{k}\right)c_1\lambda\cos(\alpha\lambda x), c_1\cos(\alpha\lambda x), ic_1\lambda\cos(\alpha\lambda x), 0\right)^T$$

and

$$\Phi(x) = \left(0, 0, -l\left(1+\frac{k_0}{k}\right)c_1 \cos(\alpha\lambda x), -il\left(1+\frac{k_0}{k}\right)c_1\lambda\cos(\alpha\lambda x), c_1\cos(\alpha\lambda x), ic_1\lambda\cos(\alpha\lambda x), 0, 0\right)^T$$

are solutions of (4.4) in cases (1.1) and (1.4), respectively. Hence, $i\lambda \notin \rho(\mathcal{A})$. Finally, (4.3) holds if and only if (4.1) holds.

Now, we need to show that

$$\sup_{|\lambda|\geq 1} \frac{1}{\lambda^4}\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{H}} < \infty. \tag{4.22}$$

Let us establish (4.22) by contradiction. Assume that (4.22) is false, then, there exist sequences $(\Phi_n)_n \subset D(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}} = 1, \quad \forall n \in \mathbb{N}, \tag{4.23}$$

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{4.24}$$

and

$$\lim_{n\to\infty} \lambda_n^4 \|(i\lambda_n I - \mathcal{A})\Phi_n\|_{\mathcal{H}} = 0. \tag{4.25}$$

Our goal is to derive that $\|\Phi_n\|_{\mathcal{H}} \to 0$ as a contradiction with (4.23). This will be established through several steps for each system by using different multipliers, where some of them were used in [1].

4.1. **Case of system** (1.1) – (1.3)**.** Let $\Phi_n = \left(\varphi_n, \widetilde{\varphi}_n, \psi_n, \widetilde{\psi}_n, w_n, \widetilde{w}_n, \theta_n\right)^T$. The limit (4.25) implies that

$$\begin{cases} \lambda_n^4\left[i\lambda_n\varphi_n - \widetilde{\varphi}_n\right] \to 0 \text{ in } H_*^1(0,1), \\ \lambda_n^4\left[i\rho_1\lambda_n\widetilde{\varphi}_n - k(\varphi_{nx}+\psi_n+lw_n)_x - lk_0(w_{nx}-l\varphi_n) + \delta\theta_{nx}\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4\left[i\lambda_n\psi_n - \widetilde{\psi}_n\right] \to 0 \text{ in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4\left[i\rho_2\lambda_n\widetilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx}+\psi_n+lw_n)\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4\left[i\lambda_n w_n - \widetilde{w}_n\right] \to 0 \text{ in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4\left[i\rho_1\lambda_n\widetilde{w}_n - k_0(w_{nx}-l\varphi_n)_x + lk(\varphi_{nx}+\psi_n+lw_n)\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4\left[i\rho_3\lambda_n\theta_n - \beta\theta_{nxx} + \delta\widetilde{\varphi}_{nx}\right] \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.26}$$

Step 1. Taking the inner product of $\lambda_n^4(i\lambda_n I - \mathcal{A})\Phi_n$ with Φ_n in $\mathcal{H}$ and using (2.3), we get

$$Re\left\langle\lambda_n^4(i\lambda_n I - \mathcal{A})\Phi_n, \Phi_n\right\rangle_{\mathcal{H}} = Re\left(i\lambda_n^5\|\Phi_n\|_{\mathcal{H}}^2 + \beta\lambda_n^4\|\theta_{nx}\|_{L^2(0,1)}^2\right) = \beta\lambda_n^4\|\theta_{nx}\|_{L^2(0,1)}^2.$$

So, (4.23) and (4.25) lead that

$$\lambda_n^2\theta_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.27}$$

Because θ_n in $\widetilde{H}_*^1(0,1)$ and thanks to Poincaré's inequality, we deduce that

$$\lambda_n^2\theta_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.28}$$

Step 2. Multiplying $(4.26)_1$, $(4.26)_3$ and $(4.26)_5$ by $\dfrac{1}{\lambda_n^5}$, and using (4.23) and (4.24), we obtain

$$\left\{\begin{array}{l}\varphi_n \longrightarrow 0 \text{ in } L^2(0,1),\\ \psi_n \longrightarrow 0 \text{ in } L^2(0,1),\\ w_n \longrightarrow 0 \text{ in } L^2(0,1).\end{array}\right. \tag{4.29}$$

Step 3. Multiplying $(4.26)_2$ by $\dfrac{1}{\lambda_n^5}$ and using (4.24), we find

$$i\rho_1\widetilde{\varphi}_n - \frac{k}{\lambda_n}\left(\varphi_{nx}+\psi_n+lw_n\right)_x - \frac{lk_0}{\lambda_n}\left(w_{nx}-l\varphi_n\right) + \frac{\delta}{\lambda_n}\theta_{nx} \longrightarrow 0 \text{ in } L^2(0,1).$$

Using (4.23), (4.24) and (4.27), we conclude that

$$\left(\frac{1}{\lambda_n}\varphi_{nxx}\right)_n \text{ is bounded in } L^2(0,1). \tag{4.30}$$

Step 4. Taking the inner product of $(4.26)_7$ with $\dfrac{i}{\lambda_n^3}\varphi_{nx}$ in $L^2(0,1)$ and using (4.23) and (4.24), we entail

$$\begin{aligned}&\rho_3\left\langle \lambda_n^2\theta_n, \varphi_{nx}\right\rangle_{L^2(0,1)} - \beta\left\langle \lambda_n\theta_{nxx}, i\varphi_{nx}\right\rangle_{L^2(0,1)}\\ &-\delta\left\langle \lambda_n\left(i\lambda_n\varphi_{nx}-\widetilde{\varphi}_{nx}\right), i\varphi_{nx}\right\rangle_{L^2(0,1)} + \delta\lambda_n^2\left\|\varphi_{nx}\right\|^2_{L^2(0,1)} \longrightarrow 0,\end{aligned}$$

then, integrating by parts and using the boundary conditions, we arrive at

$$\begin{aligned}&\rho_3\left\langle \lambda_n^2\theta_n, \varphi_{nx}\right\rangle_{L^2(0,1)} + \beta\left\langle \lambda_n^2\theta_{nx}, \frac{i}{\lambda_n}\varphi_{nxx}\right\rangle_{L^2(0,1)}\\ &-\delta\left\langle \lambda_n\left(i\lambda_n\varphi_{nx}-\widetilde{\varphi}_{nx}\right), i\varphi_{nx}\right\rangle_{L^2(0,1)} + \delta\lambda_n^2\left\|\varphi_{nx}\right\|^2_{L^2(0,1)} \longrightarrow 0.\end{aligned} \tag{4.31}$$

Combining (4.23), (4.24), $(4.26)_1$, (4.27), (4.28), (4.30) and (4.31), it follows that

$$\lambda_n\varphi_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.32}$$

Moreover, again by $(4.26)_1$, we see that

$$\widetilde{\varphi}_{nx} \to 0 \text{ in } L^2(0,1), \tag{4.33}$$

and, as φ_n, $\widetilde{\varphi}_n \in H^1_*(0,1)$ and thanks to Poincaré's inequality, we remark also

$$\lambda_n\varphi_n \longrightarrow 0 \text{ in } L^2(0,1) \tag{4.34}$$

and

$$\widetilde{\varphi}_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.35}$$

Step 5. Multiplying $(4.26)_3$ and $(4.26)_5$ by $\dfrac{1}{\lambda_n^4}$, and using (4.23) and (4.24), we have

$$\left(\lambda_n\psi_n\right)_n \text{ and } \left(\lambda_n w_n\right)_n \text{ are bounded in } L^2(0,1). \tag{4.36}$$

Step 6. Taking the inner product of $(4.26)_2$ with $\dfrac{i}{\lambda_n^3}\widetilde{\varphi}_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}&\rho_1\left\|\lambda_n\widetilde{\varphi}_n\right\|^2_{L^2(0,1)} + k\left\langle \lambda_n\left(\varphi_{nx}+\psi_n+lw_n\right), i\widetilde{\varphi}_{nx}\right\rangle_{L^2(0,1)}\\ &+lk_0\left\langle \lambda_n w_n, i\widetilde{\varphi}_{nx}\right\rangle_{L^2(0,1)} + l^2k_0\left\langle \lambda_n\varphi_n, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} + \delta\left\langle \lambda_n\theta_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \to 0.\end{aligned} \tag{4.37}$$

So, using (4.23), (4.24), (4.27), (4.32), (4.33), (4.34) and (4.36), we deduce that

$$\lambda_n\widetilde{\varphi}_n \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.38}$$

and by (4.24) and $(4.26)_1$, we find

$$\lambda_n^2 \varphi_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.39}$$

Step 7. Multiplying $(4.26)_4$ and $(4.26)_6$ by $\frac{1}{\lambda_n^5}$ and using (4.24), we obtain

$$\begin{cases} i\rho_2 \widetilde{\psi}_n - \frac{b}{\lambda_n}\psi_{nxx} + \frac{k}{\lambda_n}(\varphi_{nx} + \psi_n + lw_n) \to 0 \text{ in } L^2(0,1), \\ i\rho_1 \widetilde{w}_n - \frac{k_0}{\lambda_n} w_{nxx} + \frac{lk_0}{\lambda_n}\varphi_{nx} + \frac{lk}{\lambda_n}(\varphi_{nx} + \psi_n + lw_n) \to 0 \text{ in } L^2(0,1). \end{cases}$$

Exploiting (4.23) and (4.24), it appears that

$$\left(\frac{1}{\lambda_n}\psi_{nxx}\right)_n \text{ and } \left(\frac{1}{\lambda_n} w_{nxx}\right)_n \text{ are bounded in } L^2(0,1). \tag{4.40}$$

Step 8. Taking the inner product of $(4.26)_2$ with $\frac{1}{\lambda_n^4}[k\psi_{nx} + l(k+k_0)w_{nx}]$ in $L^2(0,1)$, we arrive at

$$\begin{aligned} &\rho_1 \left\langle i\lambda_n \widetilde{\varphi}_n, [k\psi_{nx} + l(k+k_0)w_{nx}]\right\rangle_{L^2(0,1)} - k\left\langle \varphi_{nxx}, [k\psi_{nx} + l(k+k_0)w_{nx}]\right\rangle_{L^2(0,1)} \\ &- \|k\psi_{nx} + l(k+k_0)w_{nx}\|^2_{L^2(0,1)} + l^2k_0\left\langle \varphi_n, [k\psi_{nx} + l(k+k_0)w_{nx}]\right\rangle_{L^2(0,1)} \\ &+\delta\left\langle \theta_{nx}, [k\psi_{nx} + l(k+k_0)w_{nx}]\right\rangle_{L^2(0,1)} \to 0. \end{aligned} \tag{4.41}$$

Again, integrating by parts and using the boundary conditions, we see that

$$\left\langle \varphi_{nxx}, [k\psi_{nx} + l(k+k_0)w_{nx}]\right\rangle_{L^2(0,1)} = -\left\langle \lambda_n\varphi_{nx}, \left[\frac{k}{\lambda_n}\psi_{nxx} + \frac{l(k+k_0)}{\lambda_n}w_{nxx}\right]\right\rangle_{L^2(0,1)}. \tag{4.42}$$

Then, using (4.32), (4.40) and (4.42), we deduce that

$$\left\langle \varphi_{nxx}, [k\psi_{nx} + l(k+k_0)w_{nx}]\right\rangle_{L^2(0,1)} \to 0, \tag{4.43}$$

so, exploiting (4.23), (4.24), (4.27), (4.29), (4.38), (4.41) and (4.43), we entail

$$k\psi_{nx} + l(k+k_0)w_{nx} \to 0 \text{ in } L^2(0,1). \tag{4.44}$$

Step 9. Taking the inner product of $(4.26)_4$ with $\frac{1}{\lambda_n^4}\psi_n$ in $L^2(0,1)$, using (4.23) and (4.24), integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned} &-\rho_2\left\langle \widetilde{\psi}_n, \left(i\lambda_n\psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} \\ &+b\|\psi_{nx}\|^2_{L^2(0,1)} + k\left\langle(\varphi_{nx} + \psi_n + lw_n), \psi_n\right\rangle_{L^2(0,1)} \to 0, \end{aligned}$$

then, using (4.23), (4.24), $(4.26)_3$ and (4.29), we find

$$b\|\psi_{nx}\|^2_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.45}$$

On the other hand, taking the inner product of $(4.26)_6$ with $\frac{1}{\lambda_n^4}w_n$ in $L^2(0,1)$, using (4.23) and (4.24), integrating by parts and using the boundary conditions, we observe that

$$\begin{aligned} &-\rho_1\left\langle \widetilde{w}_n, \left(i\lambda_n w_n - \widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\left\|\widetilde{w}_n\right\|^2_{L^2(0,1)} + k_0\|w_{nx}\|^2_{L^2(0,1)} \\ &-lk_0\left\langle \varphi_n, w_{nx}\right\rangle_{L^2(0,1)} + lk\left\langle(\varphi_{nx} + \psi_n + lw_n), w_n\right\rangle_{L^2(0,1)} \to 0. \end{aligned} \tag{4.46}$$

By (4.23), (4.24), $(4.26)_5$ and (4.29), it follows that

$$k_0\|w_{nx}\|^2_{L^2(0,1)} - \rho_1\left\|\widetilde{w}_n\right\|^2_{L^2(0,1)} \to 0. \tag{4.46}$$

Step 10. Taking the inner product of $(4.26)_4$ with $\frac{1}{\lambda_n^4} w_n$ and of $(4.26)_6$ with $\frac{1}{\lambda_n^4}\psi_n$, and using (4.23) and (4.24), we infer that

$$\begin{cases} \left\langle \left[i\lambda_n \rho_2 \widetilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l w_n\right)\right], w_n \right\rangle_{L^2(0,1)} \to 0, \\ \left\langle \left[i\lambda_n \rho_1 \widetilde{w}_n - k_0 \left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l w_n\right)\right], \psi_n \right\rangle_{L^2(0,1)} \to 0. \end{cases}$$

Integrating by parts and using the boundary conditions, it appears that

$$-\rho_2 \left\langle \widetilde{\psi}_n, \left(i\lambda_n w_n - \widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - \rho_2 \left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)}$$

$$+ b\left\langle \psi_{nx}, w_{nx}\right\rangle_{L^2(0,1)} + k\left\langle \left(\varphi_{nx} + \psi_n + l w_n\right), w_n \right\rangle_{L^2(0,1)} \to 0$$

and

$$-\rho_1 \left\langle \widetilde{w}_n, \left(i\lambda_n \psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_1 \left\langle \widetilde{w}_n, \widetilde{\psi}_n \right\rangle_{L^2(0,1)}$$

$$+ k_0\left\langle \left(w_{nx} - l\varphi_n\right), \psi_{nx}\right\rangle_{L^2(0,1)} + lk\left\langle \left(\varphi_{nx} + \psi_n + l w_n\right), \psi_n \right\rangle_{L^2(0,1)} \to 0,$$

then, using (4.23), (4.24), $(4.26)_3$, $(4.26)_5$ and (4.29), we obtain

$$\begin{cases} -\rho_2 \left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} + b\left\langle \psi_{nx}, w_{nx}\right\rangle_{L^2(0,1)} \to 0, \\ -\rho_1 \left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} + k_0\left\langle \psi_{nx}, w_{nx}\right\rangle_{L^2(0,1)} \to 0, \end{cases}$$

which implies that

$$\left(\frac{\rho_2}{b} - \frac{\rho_1}{k_0}\right)\left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} \to 0 \tag{4.47}$$

and

$$\left(\frac{b}{\rho_2} - \frac{k_0}{\rho_1}\right)\left\langle \psi_{nx}, w_{nx}\right\rangle_{L^2(0,1)} \to 0. \tag{4.48}$$

Step 11. At this stage, we will consider two cases.

Case 1: $\frac{b}{\rho_2} \neq \frac{k_0}{\rho_1}$. From (4.47) and (4.48), we see that

$$\left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} \to 0 \quad \text{and} \quad \left\langle \psi_{nx}, w_{nx}\right\rangle_{L^2(0,1)} \to 0. \tag{4.49}$$

Therefore, taking the inner product of (4.44), first, with ψ_{nx}, and second, with w_{nx}, we remark that

$$\psi_{nx} \to 0 \quad \text{and} \quad w_{nx} \to 0 \text{ in } L^2(0,1), \tag{4.50}$$

and then, by (4.45), (4.46) and (4.50),

$$\widetilde{\psi}_n \to 0 \quad \text{and} \quad \widetilde{w}_n \to 0 \text{ in } L^2(0,1). \tag{4.51}$$

Finally, combining (4.28), (4.29), (4.32), (4.35), (4.50) and (4.51), we get

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow 0, \tag{4.52}$$

which is a contradiction with (4.23), so (4.22) holds. Consequentely, (4.2) is satisfied.

Case 2: $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$. Multiplying $(4.26)_4$ and $(4.26)_6$ by $\frac{1}{\lambda_n}$, and using $(4.26)_3$, $(4.26)_5$ and (4.24), we obtain

$$\begin{cases} \lambda_n^3 \left[-\dfrac{\rho_2}{b} \lambda_n^2 \psi_n - \psi_{nxx} + \dfrac{k}{b} \left(\varphi_{nx} + \psi_n + l w_n \right) \right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^3 \left[-\dfrac{\rho_2}{b} \lambda_n^2 w_n - \left(w_{nx} - l \varphi_n \right)_x + \dfrac{lk}{k_0} \left(\varphi_{nx} + \psi_n + l w_n \right) \right] \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.53}$$

Multiplying $(4.53)_1$ and $(4.53)_2$ with $\dfrac{1}{\lambda_n^3}$, and using (4.24), (4.29) and (4.32), we find

$$\begin{cases} \dfrac{\rho_2}{b} \lambda_n^2 \psi_n + \psi_{nxx} \to 0 \text{ in } L^2(0,1), \\ \dfrac{\rho_2}{b} \lambda_n^2 w_n + w_{nxx} \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.54}$$

Multiplying $(4.54)_1$ by k and $(4.54)_2$ by $l(k+k_0)$ and adding the obtained limits, and multiplying $(4.54)_1$ by k and $(4.54)_2$ by $-l(k+k_0)$ and adding the limits, we entail

$$\begin{cases} \dfrac{\rho_2}{b} \lambda_n^2 \left[k\psi_n + l(k+k_0) w_n \right] + \left[k\psi_{nxx} + l(k+k_0) w_{nxx} \right] \to 0 \text{ in } L^2(0,1), \\ \dfrac{\rho_2}{b} \lambda_n^2 \left[k\psi_n - l(k+k_0) w_n \right] + \left[k\psi_{nxx} - l(k+k_0) w_{nxx} \right] \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.55}$$

Taking the inner product of $(4.55)_1$ and $(4.55)_2$ with $[k\psi_n + l(k+k_0)w_n]$, integrating by parts and using the boundary conditions, we infer that

$$\frac{\rho_2}{b} \left\| k\lambda_n \psi_n + l(k+k_0)\lambda_n w_n \right\|_{L^2(0,1)}^2 - \left\| k\psi_{nx} + l(k+k_0) w_{nx} \right\|_{L^2(0,1)}^2 \to 0$$

and

$$\begin{aligned} &\frac{\rho_2}{b} \left\langle \lambda_n^2 \left[k\psi_n - l(k+k_0) w_n \right], \left[k\psi_n + l(k+k_0) w_n \right] \right\rangle_{L^2(0,1)} \\ &- \left\langle \left[k\psi_{nx} - l(k+k_0) w_{nx} \right], \left[k\psi_{nx} + l(k+k_0) w_{nx} \right] \right\rangle_{L^2(0,1)} \to 0, \end{aligned}$$

so, using (4.23) and (4.44), it follows that

$$\begin{cases} k\lambda_n \psi_n + l(k+k_0)\lambda_n w_n \to 0 \text{ in } L^2(0,1), \\ k^2 \left\| \lambda_n \psi_n \right\|_{L^2(0,1)}^2 - l^2 (k+k_0)^2 \left\| \lambda_n w_n \right\|_{L^2(0,1)}^2 \to 0. \end{cases} \tag{4.56}$$

Taking the inner product of $(4.53)_1$ with $\dfrac{1}{\lambda_n} w_n$ and $(4.53)_2$ with $\dfrac{1}{\lambda_n} \psi_n$, using (4.23) and (4.24), integrating by parts and using the boundary conditions, we arrive at

$$\begin{aligned} &-\frac{\rho_2}{b} \lambda_n^4 \langle \psi_n, w_n \rangle_{L^2(0,1)} + \lambda_n^2 \langle \psi_{nx}, w_{nx} \rangle_{L^2(0,1)} - \frac{k}{b} \left\langle \lambda_n^2 \varphi_n, w_{nx} \right\rangle_{L^2(0,1)} \\ &+ \frac{k}{b} \langle \lambda_n \psi_n, \lambda_n w_n \rangle_{L^2(0,1)} + \frac{lk}{b} \left\| \lambda_n w_n \right\|_{L^2(0,1)}^2 \to 0 \end{aligned} \tag{4.57}$$

and

$$\begin{aligned} &-\frac{\rho_2}{b} \lambda_n^4 \langle \psi_n, w_n \rangle_{L^2(0,1)} + \lambda_n^2 \langle \psi_{nx}, w_{nx} \rangle_{L^2(0,1)} - l\left(1 + \frac{k}{k_0} \right) \left\langle \psi_{nx}, \lambda_n^2 \varphi_n \right\rangle_{L^2(0,1)} \\ &+ \frac{lk}{k_0} \left\| \lambda_n \psi_n \right\|_{L^2(0,1)}^2 + \frac{l^2 k}{k_0} \langle \lambda_n \psi_n, \lambda_n w_n \rangle_{L^2(0,1)} \to 0, \end{aligned} \tag{4.58}$$

therefore, multiplying (4.57) by $\dfrac{bk_0}{k}$, and (4.58) by $-\dfrac{bk_0}{k}$, adding the obtained limits and using (4.23) and (4.39), it appears that

$$lk_0 \left\| \lambda_n w_n \right\|_{L^2(0,1)}^2 - lb \left\| \lambda_n \psi_n \right\|_{L^2(0,1)}^2 + \left(k_0 - l^2 b \right) \langle \lambda_n \psi_n, \lambda_n w_n \rangle_{L^2(0,1)} \to 0. \tag{4.59}$$

By taking the inner product of $(4.56)_1$ with $\lambda_n \psi_n$, combining $(4.56)_2$ and (4.59), and using (4.36), we have

$$\begin{cases} k \left\| \lambda_n \psi_n \right\|^2_{L^2(0,1)} + l(k+k_0) \left\langle \lambda_n w_n, \lambda_n \psi_n \right\rangle_{L^2(0,1)} \to 0, \\ \dfrac{1}{l(k+k_0)^2} \left[k_0 k^2 - bl^2(k+k_0)^2 \right] \left\| \lambda_n \psi_n \right\|^2_{L^2(0,1)} + \left(k_0 - l^2 b \right) \left\langle \lambda_n w_n, \lambda_n \psi_n \right\rangle_{L^2(0,1)} \to 0, \end{cases} \tag{4.60}$$

so, multiplying $(4.60)_1$ by $\dfrac{(k+k_0)\left(k_0 - l^2 b\right)}{k_0}$, and $(4.60)_2$ by $-\dfrac{l\,(k+k_0)^2}{k_0}$ and adding the obtained limits, we get

$$\left[kk_0 + bl^2(k+k_0) \right] \ \left\| \lambda_n \psi_n \right\|^2_{L^2(0,1)} \to 0.$$

Thus,

$$\lambda_n \psi_n \to 0 \text{ in } L^2\left(0,1\right) \tag{4.61}$$

and, using $(4.56)_1$,

$$\lambda_n w_n \to 0 \text{ in } L^2\left(0,1\right). \tag{4.62}$$

Using (4.24), $(4.26)_3$, $(4.26)_5$, (4.61) and (4.62), we deduce that

$$\begin{cases} \widetilde{\psi}_n \to 0 \text{ in } L^2\left(0,1\right), \\ \widetilde{w}_n \to 0 \text{ in } L^2\left(0,1\right). \end{cases} \tag{4.63}$$

Taking the inner product of $(4.54)_1$ with ψ_n, and $(4.54)_2$ with w_n, integrating by parts and using the boundary conditions, we entail

$$\begin{cases} \dfrac{\rho_2}{b} \left\| \lambda_n \psi_n \right\|^2_{L^2(0,1)} - \left\| \psi_{nx} \right\|^2_{L^2(0,1)} \to 0, \\ \dfrac{\rho_2}{b} \left\| \lambda_n w_n \right\|^2_{L^2(0,1)} - \left\| w_{nx} \right\|^2_{L^2(0,1)} \to 0, \end{cases}$$

then, from (4.61) and (4.62), we find

$$\begin{cases} \psi_{nx} \to 0 \text{ in } L^2\left(0,1\right), \\ w_{nx} \to 0 \text{ in } L^2\left(0,1\right). \end{cases} \tag{4.64}$$

Finally, (4.28), (4.29), (4.32), (4.35), (4.63) and (4.64) imply (4.52), which is a contradiction with (4.23). Consequentely, in both cases $\dfrac{b}{\rho_2} \neq \dfrac{k_0}{\rho_1}$ and $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$, (4.22) holds, and hence, (4.2) in case of system $(1.1) - (1.3)$ is satisfied.

4.2. **Case of system** (1.2) − (1.5)**.** Let $\Phi_n = \left(\varphi_n, \widetilde{\varphi}_n, \psi_n, \widetilde{\psi}_n, w_n, \widetilde{w}_n, \theta_n, \widetilde{\theta}_n\right)^T$. The limit (4.25) implies that

$$\begin{cases} \lambda_n^4 \left[i\lambda_n \varphi_n - \widetilde{\varphi}_n\right] \to 0 \text{ in } H_*^1(0,1), \\ \lambda_n^4 \left[i\rho_1 \lambda_n \widetilde{\varphi}_n - k\left(\varphi_{nx} + \psi_n + lw_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) + \delta \widetilde{\theta}_{nx}\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n \psi_n - \widetilde{\psi}_n\right] \to 0 \text{ in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_2 \lambda_n \widetilde{\psi}_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + lw_n\right)\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n w_n - \widetilde{w}_n\right] \to 0 \text{ in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left[i\rho_1 \lambda_n \widetilde{w}_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + lw_n\right)\right] \to 0 \text{ in } L^2(0,1), \\ \lambda_n^4 \left[i\lambda_n \theta_n - \widetilde{\theta}_n\right] \to 0 \text{ in } H_*^1(0,1), \\ \lambda_n^4 \left[i\rho_3 \lambda_n \theta_n - \beta\theta_{nxx} - \gamma \widetilde{\theta}_{nxx} + \delta \widetilde{\varphi}_{nx}\right] \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.65}$$

Step 1. Taking the inner product of $\lambda_n^4 \left(i\,\lambda_n\, I \,-\, \mathcal{A}\right)\ \Phi_n$ with Φ_n in $\mathcal{H}$ and using (2.3), we get

$$Re \left\langle \lambda_n^4 \left(i\,\lambda_n\, I \,-\, \mathcal{A}\right)\ \Phi_n, \Phi_n \right\rangle_{\mathcal{H}} = Re \left(i\lambda_n^5 \left\|\Phi_n\right\|_{\mathcal{H}}^2 + \gamma \lambda_n^4 \left\|\widetilde{\theta}_{nx}\right\|_{L^2(0,1)}^2 \right) = \gamma \lambda_n^4 \left\|\widetilde{\theta}_{nx}\right\|_{L^2(0,1)}^2.$$

So, (4.23) and (4.25) lead to

$$\lambda_n^2 \widetilde{\theta}_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.66}$$

Because $\widetilde{\theta}_n \in \widetilde{H}_*^1(0,1)$ and thanks to Poincaré's inequality, we deduce that

$$\lambda_n^2 \widetilde{\theta}_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.67}$$

Multiplying $(4.65)_7$ by $\dfrac{1}{\lambda_n^2}$, and using (4.23), (4.24), (4.66) and (4.67), we find

$$\lambda_n^3 \theta_{nx} \longrightarrow 0 \text{ in } L^2(0,1) \tag{4.68}$$

and

$$\lambda_n^3 \theta_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.69}$$

Step 2. Multiplying $(4.65)_1$, $(4.65)_3$ and $(4.65)_5$ by $\dfrac{1}{\lambda_n^5}$, and using (4.23) and (4.24), we obtain (4.29).

Step 3. Multiplying $(4.65)_2$ by $\dfrac{1}{\lambda_n^5}$ and using (4.24), we entail

$$i\rho_1 \widetilde{\varphi}_n - \frac{k}{\lambda_n}\left(\varphi_{nx} + \psi_n + lw_n\right)_x - \frac{lk_0}{\lambda_n}\left(w_{nx} - l\varphi_n\right) + \frac{\delta}{\lambda_n} \widetilde{\theta}_{nx} \longrightarrow 0 \text{ in } L^2(0,1).$$

Using (4.23), (4.24) and (4.66), (4.30) follows.

Step 4. Taking the inner product of $(4.65)_8$ with $\dfrac{i}{\lambda_n^3}\varphi_{nx}$ in $L^2(0,1)$ and using (4.23) and (4.24), we infer that

$$\left\langle \lambda_n \left(i\rho_3 \lambda_n \theta_n - \beta\theta_{nxx} - \gamma \widetilde{\theta}_{nxx} + \delta \widetilde{\varphi}_{nx} \right), i\varphi_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

therefore, integrating by parts and using the boundary conditions, we arrive at

$$\begin{aligned}(4.70)\qquad &\rho_3 \left\langle \lambda_n^2 \theta_n, \varphi_{nx} \right\rangle_{L^2(0,1)} + \left\langle \lambda_n^2 \left(\beta \theta_{nx} + \gamma \widetilde{\theta}_{nx} \right), \frac{i}{\lambda_n} \varphi_{nxx} \right\rangle_{L^2(0,1)} \\ &-\delta \left\langle \lambda_n \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx} \right), i\varphi_{nx} \right\rangle_{L^2(0,1)} + \delta \lambda_n^2 \left\| \varphi_{nx} \right\|_{L^2(0,1)}^2 \longrightarrow 0.\end{aligned}$$

Hence, from (4.23), (4.24), (4.30), $(4.65)_1$, (4.66), (4.68), (4.69) and (4.70), we conclude (4.32). Moreover, by $(4.65)_1$, we find (4.33). As φ_n, $\widetilde{\varphi} \in H_*^1(0,1)$, we have also (4.34) and (4.35).

Step 5. Multiplying $(4.65)_3$ and $(4.26)_5$ by $\frac{1}{\lambda_n^4}$, and using (4.23) and (4.24), we find (4.36).

Step 6. Taking the inner product of $(4.65)_2$ with $\frac{i}{\lambda_n^3} \widetilde{\varphi}_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}(4.71)\qquad &\rho_1 \left\| \lambda_n \widetilde{\varphi}_n \right\|_{L^2(0,1)}^2 + k \left\langle \lambda_n \left(\varphi_{nx} + \psi_n + l w_n \right), i\widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} \\ &+lk_0 \left\langle \lambda_n w_n, i\widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} + l^2 k_0 \left\langle \lambda_n \varphi_n, i\widetilde{\varphi}_n \right\rangle_{L^2(0,1)} + \delta \left\langle \lambda_n \widetilde{\theta}_{nx}, i\widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \to 0.\end{aligned}$$

So, using (4.23), (4.32), (4.33), (4.34), (4.36), (4.66) and (4.71), we deduce (4.38). And by (4.23), (4.24) and $(4.65)_1$, we obtain (4.39).

Step 7. Exactely as in the case of system (1.1) – (1.3), step 7, using $(4.65)_4$ and $(4.65)_6$, we entail (4.40).

Step 8. As in the case of system (1.1) – (1.3), step 8, considering the inner product of $(4.65)_2$ with $\frac{1}{\lambda_n^4} \left[k\psi_{nx} + l(k+k_0) w_{nx} \right]$ in $L^2(0,1)$, integrating by parts and using the boundary conditions and (4.66) (instead of (4.27)), we get (4.44).

Step 9. The end of the proof of (4.2) in case of system (1.2) – (1.5) is the same as for system (1.1) – (1.3), step 9 - step 11. Hence, the proof of our Theorem 4.1 is completed. □

5. General comments and issues

1. In this paper, we proved the well-posedness as well as the polynomial stability and the lack of exponential stability for (1.1) – (1.3) and (1.2) – (1.5), and we obtained the polynomial decay rate of the solutions. The natural question that we can ask is whether the obtained decay rate (4.2) is optimal.

2. The second question is the extension of our results to the case of other boundary conditions than (1.3), specially the proof of the lack of exponential stability.

3. The last interesting question we note here is proving the tability of (1.1) and (1.4) in the whole space $\mathbb{R}$ (instead of $(0,1)$).

4. When the Bresse system is controled at least via the shear angle or the longitudinal displacements (that is $F_2 \neq 0$ or $F_3 \neq 0$), the exponential stability holds true under some restrictions on the coefficients; see, for example, [1], [2], [5], [9], [11], [12] and [13]. A comparaison of these results with the ones of the present paper and [10] indicates that the dissipation is better propagated to the whole system from the second or third equation of the Bresse system than from the first one. This fact can be explained by the weakness of the role played by the first equation, caused by the coupling term $(\varphi_x + \psi + lw)_x$, in comparaison with the one played by the other two equations.

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