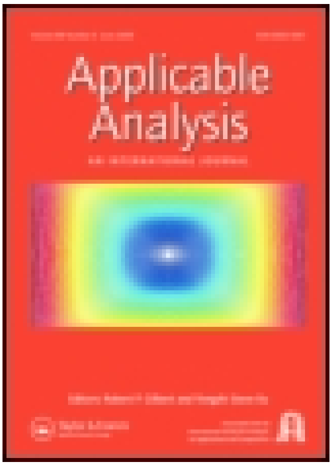

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Asymptotic behavior for coupled abstract evolution equations with one infinite memory

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Asymptotic behavior for coupled abstract evolution equations with one infinite memory

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In this paper, we consider two coupled abstract linear evolution equations with one infinite memory acting on the first equation. Our work is motivated by the recent results of [42], where the authors considered the case of two wave equations with one convolution kernel converging exponentially to zero at infinity, and proved the lack of exponential decay. On the other hand, the authors of [42] proved that the solutions decay polynomially at infinity with a decay rate depending on the regularity of the initial data. Under a boundedness condition on the past history data, we prove that the stability of our abstract system holds for convolution kernels having much weaker decay rates than the exponential one. The general and precise decay estimate of solution we obtain depends on the growth of the convolution kernel at infinity, the regularity of the initial data, and the connection between the operators describing the considered equations. We also present various applications to some distributed coupled systems such as wave-wave, Petrovsky-Petrovsky, wave-Petrovsky, and elasticity-elasticity.

Keywords: well-posedness; asymptotic behavior; memory; coupled evolution equations; semigroups theory; energy method

AMS Subject Classifications: 35L05; 35L15; 35L70; 93D15

1. Introduction

The aim of this paper is the study of the well-posedness and asymptotic behavior when time goes to infinity of solutions of the following coupled system of two linear abstract evolution equations of second-order with one infinite memory acting only on the first equation:

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds + \tilde{B}v(t) = 0, & \forall t > 0, \\ v_{tt}(t) + \tilde{A}v(t) + \tilde{B}u(t) = 0, & \forall t > 0, \end{cases} \tag{1.1}$$

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with initial conditions

$$\begin{cases} u(-t) = u_0(t), \quad \forall t \in \mathbb{R}_+, \\ v(0) = v_0, \quad u_t(0) = u_1, \quad v_t(0) = v_1, \end{cases} \tag{1.2}$$

where $A : D(A) \to H$, $\tilde{A} : D(\tilde{A}) \to H$, and $B : D(B) \to H$ are self-adjoint linear positive definite operators with domains $D(A) \subset D(B) \subset H$ and $D(\tilde{A}) \subset H$ such that the embeddings are dense and compact, $\tilde{B} : H \to H$ is a self-adjoint linear bounded operator, H is a real Hilbert space with inner product and a corresponding norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and the convolution kernel $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function. The unknown $(u, v) : \mathbb{R}_+ \to H \times H$ is the state of the system (1.1) and (1.2) corresponding to the initial data (u_0, v_0, u_1, v_1). The infinite integral in the first equation of (1.1) represents the infinite memory term which plays solely the role of dissipation for the whole system (1.1) and (1.2). The subscript t as well as $'$ denote the derivative with respect to t. The partial derivative with respect to a variable y is noted ∂_y.

Equation (1.1) can describe the dynamics of linear viscoelastic solids, a generalized Kirchhoff viscoelastic beam with memory and systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model; see, for example, [1–3], and [4] for more details concerning the physical phenomena which are modeled by differential equations with memory.

The problem of well-posedness and stability of (1.1) and (1.2) has attracted considerable attention in recent years and an important amount of research has been devoted in this direction, where diverse types of dissipative mechanisms have been introduced and several stability results have been obtained. The main objective concerning the stability in the presence of memory is determining the largest class of kernels g which guarantee the stability and the best relation between the decay rate of g and the asymptotic behavior of solutions of the considered system. Let us recall here some known results in this direction related to our goal in this paper (further results can be found in the list of references below, which is not exhaustive, and the references therein).

(A) *The uncoupled case* In the uncoupled case: $\tilde{B} = 0$, it is well known that the second equation of (1.1):

$$v_{tt}(t) + \tilde{A}v(t) = 0, \quad \forall t > 0 \tag{1.3}$$

is well-posed and it is a conservative equation; that is, the energy of (1.3) defined by

$$E_v(t) = \frac{1}{2}\left(\|v_t(t)\|^2 + \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 \right), \quad \forall t \in \mathbb{R}_+ \tag{1.4}$$

(under some assumptions on $\tilde{A}$) is a constant function: $E_v(t) = E_v(0)$, for all $t \in \mathbb{R}_+$, which means that E_v is conserved and equal to the initial energy along the trajectory of v.

Concerning the first equation of (1.1) with $\tilde{B} = 0$:

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds = 0, \quad \forall t > 0, \tag{1.5}$$

a large amount of literature is available for this model, addressing problems of the existence, uniqueness, and asymptotic behavior in time (see [2,3,5–11] and the references cited therein). The nonlinear one-dimensional viscoelastic wave equation has been investigated in [6], where it was showed that the energy of solution tends to zero asymptotically under the

Dirichlet boundary conditions, but no decay rate was given in [6]. Under some restrictions on A and B, and the condition

$$\exists \delta_1, \quad \delta_2 > 0 : -\delta_2 g(s) \leq g'(s) \leq -\delta_1 g(s), \quad \forall s \in \mathbb{R}_+, \tag{1.6}$$

the exponential decay of solutions of (1.5) (in various contexts and using different approaches) was obtained in [2,7,9], and [10]; that is, the energy of (1.5) defined by (under some assumptions on A, B and g)

$$E_u(t) = \frac{1}{2}\left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 - \left(\int_0^{+\infty} g(s)ds\right)\|B^{\frac{1}{2}}u(t)\|^2\right) + \frac{1}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}(u(t)-u(t-s))\|^2 ds, \quad \forall t \in \mathbb{R}_+ \tag{1.7}$$

satisfies

$$\exists M, \quad m > 0 : E_u(t) \leq Me^{-mt}, \quad \forall t \in \mathbb{R}_+. \tag{1.8}$$

In [5], it was proved that the weaker condition

$$\exists \delta_1 \geq 1, \quad \exists \delta_2 > 0 : g(t+s) \leq \delta_1 e^{-\delta_2 t} g(s), \quad \forall t \in \mathbb{R}_+, \text{ for a.e. } s \in \mathbb{R}_+ \tag{1.9}$$

is necessary for (1.5) to be exponentially stable. Condition (1.9) implies that g converges exponentially to zero at infinity but it does not involve the derivative of g, which allows g to have horizontal inflection points or even flat zones; that is

$$\{s \in \mathbb{R}_+ : g(s) > 0 \text{ and } g'(s) = 0\} \neq \emptyset.$$

In the particular case of the wave equation, it was proved in [11] that the exponential stability (1.8) holds if and only if g satisfies (1.9) and the set $\{s \in \mathbb{R}_+ : g'(s) < 0\}$ has positive Lebesgue measure. Equation (1.5) with $B = A^\alpha$, $\alpha \in [0, 1[$ and g satisfies (1.6) was considered in [3], where the authors proved that, for large t,

$$\exists M > 0 : E_u(t) \leq M\left(\frac{\ln t}{t}\right)^{\frac{1}{2-2\alpha}} \ln t,$$

and the decay rate is optimal in the sense that $t^{\frac{-1}{2-2\alpha}}$ cannot be improved. The question concerning the stability of (1.5) with g having a general growth at infinity was considered in [8], where general decay estimates on E_u depending on g were established under some restrictions on A and B, and the condition (3.7) below, which implies that

$$\{s \in \mathbb{R}_+ : g(s) > 0 \text{ and } g'(s) = 0\} = \emptyset$$

but it is much weaker than (1.6) because it allows a much larger class of kernels g, where the decay rate of g at infinity can be arbitrary close to $\frac{1}{t}$; see Section 3.2 below, and the examples given in [8,12] and [13]. On the other hand, the results of [8] improve, in some particular cases, many results in the literature by obtaining a stronger and precise decay rate of E_u, and the approach of [8] can be applied to many other systems with infinite memory; see [8,12] and [13].

When the infinite integral $\int_0^{+\infty}$ is replaced by the finite one $\int_0^t$, (1.5) takes the form

$$u_{tt}(t) + Au(t) - \int_0^t g(s)Bu(t-s)ds = 0, \quad \forall t > 0 \tag{1.10}$$

whose stability issue has received considerable attention and there is now a large literature on this subject, where various decay estimates were obtained depending on the growth of g at infinity. For the viscoelastic equation, a new approach was introduced and developed in [14] and [15] to get a general estimate of stability for kernels satisfying

$$g'(s) \leq -\xi(s)g(s), \quad \forall s \in \mathbb{R}_+, \tag{1.11}$$

where ξ is a positive and nonincreasing function. Later, the approach of [14] and [15] has been applied in [16] for the Timoshenko systems with finite memory and equal speeds of wave propagation, in [17] for the Timoshenko systems with infinite memory, in [18] for a nonlinear system of viscoelastic wave equations with source terms, and in [19] for an abstract system with infinite memory. The decay results in [16–18] and [19] improve earlier ones in the literature in which only the exponential and polynomial decay rates were obtained under the following stronger condition than (1.11):

$$\exists \delta > 0, \quad \exists p \in [1, \frac{3}{2}[: \quad g'(s) \leq -\delta g^p(s), \quad \forall s \in \mathbb{R}_+; \tag{1.12}$$

see, for example, [20] and [21] in case of exponential stability under (1.12) with $p = 1$, and [22,23], and [24] in case of polynomial stability under (1.12) with $1 < p < \frac{3}{2}$. The case of Timoshenko systems with finite memory and different speeds of wave propagation was studied in [25] for kernels satisfying

$$g'(s) \leq -\xi(s)g^p(s), \quad \forall s \in \mathbb{R}_+, \tag{1.13}$$

where ξ is a positive and nonincreasing function and $p \geq 1$. We mention also the recent results in [26] and [27], where general and sufficient conditions under which the solution of (1.10) converges to zero at least as fast as the kernel at infinity were given by assuming the following condition:

$$g'(s) \leq -H(g(s)), \quad \forall s \in \mathbb{R}_+, \tag{1.14}$$

where H is a nonnegative function satisfying some hypotheses. The general relations between the decay rate of the energy and that of g obtained in [26] and [27] hold without imposing restrictive assumptions on the behavior of g at infinity.

In case of wave equations, an approach based on the integral condition

$$g(t-s) \geq \mu(t) \int_t^{+\infty} g(\tau - s)d\tau, \quad \forall t \in \mathbb{R}_+, \ \forall s \in [0, t], \tag{1.15}$$

where μ is a positive function, was introduced and developed in [28–35] for (1.10). This approach allows to deal with some arbitrary decaying kernels without assuming explicit conditions on their derivative.

(B) *The coupled case* In the coupled case: $\tilde{B} \neq 0$, the stability of (1.1) and (1.2) is more complicated since only the first equation in (1.1) is directly controlled by an infinite memory. The first and principal question which can be asked here is the following: is it possible for the unique memory term considered only on the first equation in (1.1) to stabilize the whole system (1.1) and (1.2), where the second equation in (1.1) is partially and indirectly controlled via the behavior of the first one and the coupling operator $\tilde{B}$, and in a such case, what is the relation between, in particular, the growth of g at infinity and the decay rate of the energy of (1.1) and (1.2) defined by (under some assumptions on A, $\tilde{A}$, B, $\tilde{B}$ and g)

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2\right) + \left\langle \tilde{B}u(t), v(t)\right\rangle + \frac{1}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}(u(t)-u(t-s))\|^2 ds, \quad \forall t \in \mathbb{R}_+, \quad (1.16)$$

where $g_0 = \int_0^{+\infty} g(s)ds$? The concept of indirect stability for coupled systems was introduced, as far as we know, in [36], where the controlled equation plays the role of stabilizer for the second one via the coupling terms. See [37] for further related stability results for coupled systems.

When the infinite memory term in (1.1) is replaced by the frictional damping Bu_t, and $\tilde{B} = \alpha Id$, where Id is the identity operator and $\alpha \in \mathbb{R}_+^*$, it has been proved in [38] that

$$\begin{cases} u_{tt}(t) + Au(t) + Bu_t(t) + \alpha v(t) = 0, & \forall t > 0, \\ v_{tt}(t) + \tilde{A}v(t) + \alpha u(t) = 0, & \forall t > 0 \end{cases} \quad (1.17)$$

is not exponentially stable and the asymptotic behavior of solutions is at least of polynomial type with decay rates depending on the smoothness of initial data. The results of [38] show that if the solution of (1.17) satisfies any stability estimate, then such an estimate is necessarily weaker than the exponential one. The method introduced and developed in [38] is based on a general estimate on the asymptotic behavior of solutions in terms of higher order initial energies. Some extensions of the results of [38] to the nonlinear and nondissipative cases are given in [39] and [40] in the particular case of coupled wave equations. Recently, the stability of (1.17) was considered in [41] in the particular case of coupled Euler–Bernoulli and wave equations, and with clamped boundary conditions for the Euler–Bernoulli equation. The decay estimates obtained in [41] are of polynomial type with decay rates weaker than the ones obtained in [38], but the abstract framework introduced in [38] does not include the case considered in [41], since the condition

$$\exists j \in \mathbb{N} \quad \text{and} \quad j \geq 2 : D(\tilde{A}^{\frac{j}{2}}) \subset D(A) \quad (1.18)$$

under which the results of [38] hold is not satisfied due to the clamped boundary conditions for the Euler–Bernoulli equation. See also the references of [38–40], and [41] for further existing results related to the stability of (1.17).

Concerning the problem of stability of (1.1) and (1.2), there are very few results in literature and, as far as we know, the unique results in this direction are the recent ones obtained in [42] in the particular case of wave equations:

$$\begin{cases} u_{tt}(t) - \Delta u(t) + \int_0^{+\infty} g(s)\Delta u(t-s)ds + \alpha v(t) = 0, & \forall t > 0, \\ v_{tt}(t) - \Delta v(t) + \alpha u(t) = 0, \quad \forall t > 0 \end{cases} \quad (1.19)$$

with Dirichlet boundary conditions. More precisely, the lack of exponential stability for (1.19) was proved and a polynomial decay estimate, similar to the one of [38], was obtained under the condition (1.6) and

$$\exists \delta_3 > 0 : \quad |g''(s)| \leq -\delta_3 g(s), \quad \forall s \in \mathbb{R}_+. \quad (1.20)$$

In fact, the conditions (1.6) and (1.20) are too restrictive and they imply that, in particular, g converges exponentially to zero at infinity.

Our objectives in this paper are the following:

(1) *Well-posedness (Section 2)* Following a method devised in the pioneering paper [6] to treat the memory term by considering a new auxiliary variable, we first formulate the system (1.1) and (1.2) in an abstract linear first-order system. Then, using the semigroups approach (see [43,44] and [45]), we prove the global existence, uniqueness, and smoothness of solutions of (1.1) and (1.2), where the regularity of the solution (u, v) depends on the one of the initial data (u_0, v_0, u_1, v_1).

(2) *Stability (Section 3)* Our second and main objective is proving that the stability of (1.1) and (1.2) holds for the much wider class of g satisfying assumption (A8) below. We give the decay rate of solutions of (1.1) and (1.2) explicitly in terms of the growth of g at infinity, the arbitrary regularity of the initial data (u_0, v_0, u_1, v_1), and the connection between the operators A, $\tilde{A}$, and B. This will generalize and improve several results in the literature, such as the ones of [42] in which only the case of (1.19) was considered under the much more restrictive conditions (1.6) and (1.20).

(3) *Applications (Section 4)* The abstract system (1.1) and (1.2) includes various coupled systems with only one infinite memory, where the well-posedness and stability results of Sections 2 and 3 hold. To illustrate this fact, we present the examples of wave-wave, Petrovsky-Petrovsky, wave-Petrovsky, and elasticity-elasticity systems.

2. Well-posedness

We state in this section some assumptions on A, $\tilde{A}$, B, $\tilde{B}$, and g, and give a brief proof of the global existence, uniqueness, and smoothness of solutions of (1.1) and (1.2). We assume that

(A1) There exist positive constants a_0 and a_1 satisfying

$$D(A) \subset D(B) \quad \text{and} \quad a_1\|w\|^2 \leq \|B^{\frac{1}{2}}w\|^2 \leq a_0\|A^{\frac{1}{2}}w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}). \tag{2.1}$$

(A2) There exists a positive constant $\tilde{a}_1$ satisfying

$$\tilde{a}_1\|w\|^2 \leq \|\tilde{A}^{\frac{1}{2}}w\|^2, \quad \forall w \in D(\tilde{A}^{\frac{1}{2}}). \tag{2.2}$$

(A3) The kernel g is of class $C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, nonincreasing and satisfies

$$g_0 := \int_0^{+\infty} g(s)ds < \frac{1}{a_0}. \tag{2.3}$$

Moreover, there exists a positive constant δ_0 such that

$$-g'(s) \leq \delta_0 g(s), \quad \forall s \in \mathbb{R}_+. \tag{2.4}$$

(A4) There exists a positive constant b_1 satisfying

$$\|\tilde{B}w\|^2 \leq b_1\|w\|^2, \quad \forall w \in H \tag{2.5}$$

and

$$b_1 < \sqrt{\frac{a_1 \tilde{a}_1 (1 - a_0 g_0)}{a_0}}. \tag{2.6}$$

Remark 2.1

(1) In Section 4, various applications will be presented with specific operators A, $\tilde{A}$, B, and $\tilde{B}$ satisfying hypotheses (A1), (A2), and (A4) as well as the additional ones (A5)–(A7) considered in Section 3 to get the stability of (1.1) and (1.2).

(2) Condition (2.4) implies that

$$g(s) \geq g(0) e^{-\delta_0 s}, \quad \forall s \in \mathbb{R}_+, \tag{2.7}$$

which means that the asymptotic behavior of g is at most of exponential type. Then, hypothesis (A3) includes a very wide class of functions, which can converge exponentially to zero at infinity like $g_1(s) = p_1 e^{-q_1 s}$ ($p_1, q_1 > 0$) or at a slower rate like $g_2(s) = \frac{p_2}{(1+s)^{q_2}}$ and $g_3(s) = p_3 e^{-q_3 (\ln(1+s))^{r_3}}$ ($p_2, p_3, q_3 > 0$ and $q_2, r_3 > 1$).

Following a method devised in [6] to treat the memory term by considering a new auxiliary variable η, we will formulate the system (1.1) and (1.2) in the following abstract linear first-order system:

$$\begin{cases} \mathscr{U}_t(t) = \mathscr{A}\mathscr{U}(t), & \forall t > 0, \\ \mathscr{U}(0) = \mathscr{U}_0, \end{cases} \tag{2.8}$$

where $\mathscr{U} = (u, v, u_t, v_t, \eta)^T$, $\mathscr{U}_0 = (u_0(0), v_0, u_1, v_1, \eta_0)^T \in \mathscr{H}$,

$$\mathscr{H} = D(A^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{1}{2}}) \times H \times H \times L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})),$$

$$L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) = \left\{ w : \mathbb{R}_+ \to D(B^{\frac{1}{2}}), \quad \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} w(s)\|^2 ds < +\infty \right\},$$

$$\begin{cases} \eta(t, s) = u(t) - u(t - s), & \forall t, s \in \mathbb{R}_+, \\ \eta_0(s) = \eta(0, s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+ \end{cases} \tag{2.9}$$

and $\mathscr{A}$ is a linear operator given by

$$\mathscr{A} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ (-A + g_0 B) w_1 - \int_0^{+\infty} g(s) B w_5(s) ds - \tilde{B} w_2 \\ -\tilde{A} w_2 - \tilde{B} w_1 \\ -\partial_s w_5 + w_3 \end{pmatrix} \tag{2.10}$$

with domain $\mathscr{D}(\mathscr{A})$ given by

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{l} W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{H},\ \partial_s w_5 \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}})),\ w_4 \in D(\tilde{A}^{\frac{1}{2}}), \\ w_3 \in D(A^{\frac{1}{2}}),\ w_2 \in D(\tilde{A}),\ (A - g_0 B)w_1 + \int_0^{+\infty} g(s) B w_5(s) ds \in H,\ w_5(0) = 0 \end{array} \right\}. \tag{2.11}$$

We use the classical notations $\mathscr{D}(\mathscr{A}^0) = \mathscr{H}$, $\mathscr{D}(\mathscr{A}^1) = \mathscr{D}(\mathscr{A})$ and

$$\mathscr{D}(\mathscr{A}^{n+1}) = \left\{ W \in \mathscr{D}(\mathscr{A}^n) : \mathscr{A} W \in \mathscr{D}(\mathscr{A}^n) \right\}, \quad n = 1, 2, \cdots .$$

The spaces $L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ and $\mathscr{H}$ are endowed with the inner products

$$\langle w_1, w_2 \rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \int_0^{+\infty} g(s) \left\langle B^{\frac{1}{2}} w_1(s), B^{\frac{1}{2}} w_2(s) \right\rangle ds$$

and

$$\begin{aligned} &\left\langle (w_1, w_2, w_3, w_4, w_5)^T, (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5)^T \right\rangle_{\mathscr{H}} \\ &= \left\langle A^{\frac{1}{2}} w_1, A^{\frac{1}{2}} \tilde{w}_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}} w_1, B^{\frac{1}{2}} \tilde{w}_1 \right\rangle \\ &\quad + \left\langle \tilde{A}^{\frac{1}{2}} w_2, \tilde{A}^{\frac{1}{2}} \tilde{w}_2 \right\rangle + \left\langle \tilde{B} w_2, \tilde{w}_1 \right\rangle + \left\langle \tilde{B} w_1, \tilde{w}_2 \right\rangle \\ &\quad + \langle w_3, \tilde{w}_3 \rangle + \langle w_4, \tilde{w}_4 \rangle + \langle w_5, \tilde{w}_5 \rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))} . \end{aligned}$$

Note that, from (2.3) and (2.6), one can choose $\epsilon_0 \in]\frac{b_1}{\tilde{a}_1}, \frac{(1-a_0 g_0)a_1}{b_1 a_0}[$ and, consequently,

$$\min \left\{ 1 - a_0 g_0 - \frac{\epsilon_0 b_1 a_0}{a_1}, 1 - \frac{b_1}{\epsilon_0 \tilde{a}_1} \right\} > 0.$$

Then, thanks to (2.1), (2.2) and (2.5) and the Cauchy-Schwarz and Young's inequalities, we have, for any $(w_1, w_2) \in D(A^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{1}{2}})$,

$$\begin{aligned} -g_0 \|B^{\frac{1}{2}} w_1\|^2 + \left\langle \tilde{B} w_2, w_1 \right\rangle + \left\langle \tilde{B} w_1, w_2 \right\rangle &\geq -a_0 g_0 \|A^{\frac{1}{2}} w_1\|^2 - 2\sqrt{b_1} \|w_1\| \|w_2\| \\ &\geq -a_0 g_0 \|A^{\frac{1}{2}} w_1\|^2 - b_1 \epsilon_0 \|w_1\|^2 - \frac{b_1}{\epsilon_0} \|w_2\|^2 \\ &\geq - \left(a_0 g_0 + \frac{\epsilon_0 b_1 a_0}{a_1} \right) \|A^{\frac{1}{2}} w_1\|^2 - \frac{b_1}{\epsilon_0 \tilde{a}_1} \|\tilde{A}^{\frac{1}{2}} w_2\|. \end{aligned}$$

Therefore, for $c_0 := \min \left\{ 1 - a_0 g_0 - \frac{\epsilon_0 b_1 a_0}{a_1}, 1 - \frac{b_1}{\epsilon_0 \tilde{a}_1} \right\}$,

$$\begin{aligned} c_0 \left(\|A^{\frac{1}{2}} w_1\|^2 + \|\tilde{A}^{\frac{1}{2}} w_2\|^2 \right) \leq \|A^{\frac{1}{2}} w_1\|^2 - g_0 \|B^{\frac{1}{2}} w_1\|^2 + \|\tilde{A}^{\frac{1}{2}} w_2\|^2 \\ + \left\langle \tilde{B} w_2, w_1 \right\rangle + \left\langle \tilde{B} w_1, w_2 \right\rangle . \end{aligned} \tag{2.12}$$

Consequently, $\mathscr{H}$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ is a Hilbert space and $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding.

Now, keeping (2.9) in mind, we find

$$\begin{cases} \partial_t \eta(t,s) + \partial_s \eta(t,s) = u_t(t), & \forall t,\ s \in \mathbb{R}_+, \\ \eta(t,0) = 0, & \forall t \in \mathbb{R}_+. \end{cases} \tag{2.13}$$

Therefore, we deduce from (2.10) and (2.13) that (1.1) and (1.2) is equivalent to (2.8), where the well-posedness is ensured by the following theorem:

THEOREM 2.2 *Assume that* (A1)–(A4) *hold. Then, for any* $n \in \mathbb{N}$ *and* $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^n)$, *the system* (2.8) *has a unique solution*

$$\mathscr{U} \in \cap_{k=0}^{n} C^k \left(\mathbb{R}_+, \mathscr{D}(\mathscr{A}^{n-k})\right). \tag{2.14}$$

Proof By proving that the operator $-\mathscr{A}$ is maximal monotone, Theorem 2.2 is a consequence of the semigroups theory (see [43] and [45]). So, for any $W = (w_1, w_2, w_3, w_4, w_5) \in \mathscr{D}(\mathscr{A})$, we have

$$\begin{aligned} \langle \mathscr{A} W, W \rangle_{\mathscr{H}} = & \left\langle A^{\frac{1}{2}} w_3, A^{\frac{1}{2}} w_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}} w_3, B^{\frac{1}{2}} w_1 \right\rangle + \left\langle \tilde{A}^{\frac{1}{2}} w_4, \tilde{A}^{\frac{1}{2}} w_2 \right\rangle \\ & + \left\langle \tilde{B} w_4, w_1 \right\rangle + \left\langle \tilde{B} w_3, w_2 \right\rangle + \left\langle -\tilde{A} w_2 - \tilde{B} w_1, w_4 \right\rangle \\ & + \left\langle (-A + g_0 B) w_1 - \int_0^{+\infty} g(s) B w_5(s) ds - \tilde{B} w_2, w_3 \right\rangle \\ & + \langle -\partial_s w_5 + w_3, w_5 \rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))}. \end{aligned} \tag{2.15}$$

It is clear that by the definitions of $A^{\frac{1}{2}}$, $\tilde{A}^{\frac{1}{2}}$, and $B^{\frac{1}{2}}$, and the fact that H is a real Hilbert space,

$$\langle (-A + g_0 B) w_1, w_3 \rangle = -\left\langle A^{\frac{1}{2}} w_3, A^{\frac{1}{2}} w_1 \right\rangle + g_0 \left\langle B^{\frac{1}{2}} w_3, B^{\frac{1}{2}} w_1 \right\rangle,$$

$$\left\langle -\tilde{A} w_2, w_4 \right\rangle = -\left\langle \tilde{A}^{\frac{1}{2}} w_4, \tilde{A}^{\frac{1}{2}} w_2 \right\rangle$$

and

$$\left\langle - \int_0^{+\infty} g(s) B w_5(s) ds, w_3 \right\rangle = -\langle w_3, w_5 \rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))}.$$

On the other hand, integrating by parts and using the fact that $\lim_{s \to +\infty} g(s) B^{\frac{1}{2}} w_5(s) = 0$ (due to (A3)) and $w_5(0) = 0$ (definition of $\mathscr{D}(\mathscr{A})$), we find

$$\left\langle -\frac{\partial w_5}{\partial s}, w_5 \right\rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \frac{1}{2} \int_0^{+\infty} g'(s) \|B^{\frac{1}{2}} w_5(s)\|^2 ds.$$

Consequently, inserting these four equalities in (2.15), we get

$$\langle \mathscr{A} W, W \rangle_{\mathscr{H}} = \frac{1}{2} \int_0^{+\infty} g'(s) \|B^{\frac{1}{2}} w_5(s)\|^2 ds, \tag{2.16}$$

which implies that

$$\langle \mathscr{A} W, W \rangle_{\mathscr{H}} \leq 0, \tag{2.17}$$

since g is nonincreasing. This means that $\mathscr{A}$ is dissipative. Note that, thanks to (2.4) and the fact that $w_5 \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$,

$$\left| \int_0^{+\infty} g'(s)\|B^{\frac{1}{2}} w_5(s)\|^2 ds \right| = -\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}} w_5(s)\|^2 ds \leq \delta_0 \int_0^{+\infty} g(s)\|B^{\frac{1}{2}} w_5(s)\|^2 ds < +\infty, \tag{2.18}$$

so the integral in (2.16) is well defined.

Next, we shall prove that $Id - \mathscr{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathscr{H}$, we show that there exists $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{D}(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})W = F. \tag{2.19}$$

We note that the first and second equations in (2.19) give

$$w_3 = w_1 - f_1 \quad \text{and} \quad w_4 = w_2 - f_2. \tag{2.20}$$

The last equation in (2.19) with $w_5(0) = 0$ has a unique solution

$$w_5(s) = \left(\int_0^s e^y (f_5(y) + w_1 - f_1) dy\right) e^{-s}. \tag{2.21}$$

On the other hand, plugging (2.20) and (2.21) into the third and fourth equations in (2.19), we get

$$\begin{cases} (A - g_1 B + Id)\, w_1 + \tilde{B} w_2 = \tilde{f}_1, \\ \tilde{B} w_1 + (\tilde{A} + Id) w_2 = \tilde{f}_2, \end{cases} \tag{2.22}$$

where $g_1 = \int_0^{+\infty} g(s)e^{-s} ds$, $\tilde{f}_2 = f_2 + f_4$ and

$$\tilde{f}_1 = f_1 + f_3 + \int_0^{+\infty} g(s)e^{-s} \left(\int_0^s e^y B(f_1 - f_5(y)) dy\right) ds.$$

Since $1 - a_0 g_1 > 1 - a_0 g_0 > 0$ (thanks to (2.3)) and using (2.12), the operator $\mathscr{L}$: $D(A) \times D(\tilde{A}) \to H \times H$ defined by

$$\mathscr{L} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} (A - g_1 B + Id) w_1 + \tilde{B} w_2 \\ \tilde{B} w_1 + (\tilde{A} + Id) w_2 \end{pmatrix}$$

is self-adjoint linear positive definite. Then, applying the Lax–Milgram theorem and classical regularity arguments, (2.22) has a unique solution $(w_1, w_2)^T \in D(A^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{1}{2}})$. Furthermore, coming back to (2.19) and recalling (2.20) and (2.21), we see that $W \in \mathscr{D}(\mathscr{A})$ satisfying

$$(A - g_0 B) w_1 + \int_0^{+\infty} g(s) B w_5(s) ds \in H.$$

Hence, $Id - \mathscr{A}$ is surjective. Finally, (2.17) and (2.19) mean that $-\mathscr{A}$ is a maximal monotone operator. Therefore, using Lummer–Phillips theorem (see [45]), we deduce that $\mathscr{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathscr{H}$, and then the result of Theorem 2.2 is ensured by the semigroups theory (see [43] and [45]). □

3. Asymptotic behavior

This section is devoted to the study of the asymptotic behavior of solutions of (2.8).

3.1. *Additional assumptions and stability estimate*

We start by considering the following additional assumptions:

(A5) There exist a positive constant a_2 and $j_0 \in \{0, 1\}$ such that

$$\|A^{\frac{1}{2}} w\|^2 \leq a_2 \|A^{\frac{j_0}{2}} B^{\frac{1}{2}} w\|^2, \quad \forall w \in D(A^{\frac{1}{2}} B^{\frac{j_0}{2}}). \tag{3.1}$$

(A6) There exists a positive constant $\tilde{a}_2$ such that

$$D(\tilde{A}) \subset D(A) \quad \text{and} \quad \|A^{\frac{1}{2}} w\|^2 \leq \tilde{a}_2 \|\tilde{A}^{\frac{1}{2}} w\|^2, \ \forall w \in D(\tilde{A}^{\frac{1}{2}}). \tag{3.2}$$

(A7) The constant b_1 defined in (A4) satisfies also

$$b_1 < \frac{a_1 \tilde{a}_1 (1 - a_0 g_0)}{a_0}, \tag{3.3}$$

and there exists a positive constant b_0 satisfying

$$\left\langle \tilde{B} w, w \right\rangle \geq b_0 \|w\|^2, \ \forall w \in H \tag{3.4}$$

or

$$\left\langle \tilde{B} w, w \right\rangle \leq -b_0 \|w\|^2, \ \forall w \in H. \tag{3.5}$$

(A8) The function g satisfies $g(0) > 0$ and there exists a positive constant δ such that

$$g'(s) \leq -\delta g(s), \quad \forall s \in \mathbb{R}_+ \tag{3.6}$$

or there exists an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying $G(0) = G'(0) = 0$, $\lim_{t \to +\infty} G'(t) = +\infty$ and

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{3.7}$$

Now, we introduce two sets of initial data $\mathcal{U}_0$ for which our stability estimate holds. Let, for $n \in \mathbb{N}^*$,

$$\mathcal{K}_n = \left\{ \mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n) : A^{\frac{j_0}{2}} \mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n) \right\} \tag{3.8}$$

when (3.6) holds, and

$$\mathcal{K}_n = \left\{ \mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n) : A^{\frac{j_0}{2}} \mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n) \quad \text{and} \right. \tag{3.9}$$
$$\left. \sup_{t \in \mathbb{R}_+} \max_{k=0,\cdots,n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \left\| A^{\frac{j_0}{2}} B^{\frac{1}{2}} \partial_s^k u_0(s-t) \right\|^2 ds < +\infty \right\}$$

when (3.7) holds and (3.6) does not hold.

THEOREM 3.1 *Assume that* (A1)–(A8) *hold and let* $n \in \mathbb{N}^*$ *and* $\mathcal{U}_0 \in \mathcal{K}_{i_0 n}$, *where* $i_0 = 2$ *if* $\tilde{A} = A - g_0 B$, *and* $i_0 = 3$ *if* $\tilde{A} \neq A - g_0 B$. *Then there exists a positive constant* c_n *such that*

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leq c_n G_n\left(\frac{c_n}{t}\right), \quad \forall t > 0, \tag{3.10}$$

where

$$G_0(s) = \begin{cases} s & \text{if (3.6) holds,} \\ sG'(s) & \text{if (3.7) holds and (3.6) does not hold,} \end{cases} \tag{3.11}$$

$G_1(s) = G_0^{-1}(s)$ *and* $G_m(s) = G_1(sG_{m-1}(s))$, *for* $m = 2, 3, \cdots, n$ *and* $s \in \mathbb{R}_+$.

3.2. *Comments and examples*

We give here some comments and explicit examples on the assumptions considered in this paper and the obtained stability result.

(1) The class of functions satisfying (A3) and (A8) is very wide and contains the ones which converge to zero exponentially (conditions (2.4) and (3.6)) or at a slower rate (conditions (2.4) and (3.7)) like, respectively, $g_1(s) = d_1 e^{-q_1 s}$ and $g_2(s) = \frac{d_2}{(1+s)^{q_2}}$, $d_1, q_1, d_2 > 0$ and $q_2 > 1$, where (A3) is satisfied by g_1 and g_2 provided that d_1 and d_2 are small enough so that $d_1 < \frac{q_1}{a_0}$ and $d_2 < \frac{q_2-1}{a_0}$ (thus $\int_0^{+\infty} g_1(s)ds = \dfrac{d_1}{q_1} < \dfrac{1}{a_0}$ and $\int_0^{+\infty} g_2(s)ds = \dfrac{d_2}{q_2-1} < \dfrac{1}{a_0}$), g_1 satisfies (3.6) with $\delta = q_1$, and g_2 satisfies (3.7) with $G(s) = s^p$, for any $p > \dfrac{q_2+1}{q_2-1}$. Condition (3.6) can be seen as the limit of (3.7) when G approaches Id; that is, when the decay rate of g approaches the exponential one.

(2) The assumptions (1.6) and (1.20) considered in [42] are too restrictive because they imply, in particular, that

$$g(0)e^{-\delta_2 s} \leq g(s) \leq g(0)e^{-\delta_1 s}, \quad \forall s \in \mathbb{R}_+. \tag{3.12}$$

(3) Keeping (A1) in mind, (3.1) with $j_0 = 0$ implies that A and B are equivalent. Otherwise, (3.1) with $j_0 = 1$ means that A is stronger than B. Similarly, (3.2) implies that $\tilde{A}$ is stronger than A and B.

(4) Assumptions (3.4) and (3.5) imply that the coupling operator $\tilde{B}$ is effective along the space H. This fact guarantees the control of the second equation of (1.1) via the behavior of the first one and $\tilde{B}$.

(5) In the particular case of (1.19) and under assumptions (1.6) and (1.20), the decay estimate obtained in [42] is the following:

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leq c_n t^{-n}, \quad \forall t > 0, \tag{3.13}$$

for any $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^{2n})$. Estimate (3.13) is identical to (3.10) in case (3.6), and it is stronger than (3.10) in case (3.7). The same estimate (3.10) was proved in [13] for some Timoshenko-type systems under the boundedness condition

$$\sup_{s\in\mathbb{R}_+} \max_{k=0,\cdots,n} \left\| \partial_x \partial_s^k \eta_0(s) \right\|_{L^2(]0,L[)} < +\infty. \tag{3.14}$$

On the other hand, under (3.1) with $j_0 = 1$, and the boundedness condition

$$\sup_{s\in\mathbb{R}_+} \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} u_0(s) \right\| < +\infty, \tag{3.15}$$

it was proved in [8] that the energy E_u defined in (1.7) for (1.5) satisfies, for some positive constants ϵ_0, δ_1 and δ_2, and $G_0(s) = sG'(\epsilon_0 s)$,

$$E_u(t) \le \delta_2 G_0^{-1}\left(\frac{\delta_1}{t}\right), \quad \forall t > 0. \tag{3.16}$$

The estimate (3.10) proves that (3.14) and (3.15) are not needed to get the asymptotic stability $\lim_{t\to+\infty} \|\mathcal{U}(t)\|^2_{\mathcal{H}} = 0$ (see the examples (3.18) and (3.20) below).

(6) Let us consider an example to illustrate how the smoothness of $\mathcal{U}_0$ improves the decay rate in (3.10). Let $q > 1$, $a > 0$ and $g(s) = a(1+s)^{-q}$ such that a is small enough so that (2.3) holds. Assumptions (1.6) and (1.20) considered in [42] are not satisfied, but condition (3.7) holds with $G(s) = s^p$, for any $p > \frac{q+1}{q-1}$. Then we find $G_n(s) = cs^{p_n}$, where c is a positive constant and $p_n = \sum_{m=1}^n p^{-m}$. Therefore, (3.10) gives

$$\|\mathcal{U}(t)\|^2_{\mathcal{H}} \le c_n t^{-p_n}, \quad \forall t > 0,\ \forall p > \frac{q+1}{q-1}. \tag{3.17}$$

Note that t^{-p_n} approaches t^{-n} (which is the decay rate in (3.13)) as p goes to 1^+ (that is when q converges to $+\infty$). Estimate (3.17) holds for initial data satisfying, for example,

$$\max_{k=0,\cdots,i_0 n} \left\| A^{\frac{j_0}{2}} B^{\frac{1}{2}} \partial_s^k u_0(s) \right\|^2 \le q_1(1+s)^{q_0}, \quad \forall s \in \mathbb{R}_+, \tag{3.18}$$

where q_1 is positive constant and $q_0 < \dfrac{(p-1)(q-1)-2}{p}$, so $\mathcal{U}_0 \in \mathcal{K}_{i_0 n}$. If

$$0 < q_0 < \frac{(p-1)(q-1)-2}{p},$$

the initial data satisfying (3.18) do not necessarly satisfy neither (3.14) nor (3.15).

(7) Let us consider here another example to illustrate how the estimate (3.10) generalizes and improves the ones known in literature. Let $q > 1$, d, $a > 0$ and $g(s) = ae^{-d(\ln(1+s))^q}$ such that a is small enough so that (2.3) holds. Assumptions (1.6) and (1.20) considered in [42] are not satisfied, but condition (3.7) holds with $G(s) = s^p$, for any $p > 1$. Then $G_n(s) = cs^{p_n}$, where c is a positive constant and $p_n = \sum_{m=1}^n p^{-m}$. Therefore, (3.10) gives

$$\|\mathcal{U}(t)\|^2_{\mathcal{H}} \le c_n t^{-p_n}, \quad \forall t > 0,\ \forall p > 1. \tag{3.19}$$

The decay rate t^{-p_n} in (3.19) is arbitrary close to the one of (3.13) (since p_n converges to n as p goes to 1^+) even though g does not satisfy neither (1.6) nor (1.20). Estimate (3.19) holds for initial data satisfying, for example,

$$\max_{k=0,\cdots,i_0 n} \left\| A^{\frac{j_0}{2}} B^{\frac{1}{2}} \partial_s^k u_0(s) \right\|^2 \le d_1 e^{d_2(\ln(2+s))^{q_0}}, \quad \forall s \in \mathbb{R}_+, \tag{3.20}$$

where d_1 and d_2 are positive constants, and $q_0 < q$, so $\mathscr{U}_0 \in \mathscr{K}_{ion}$. If $0 < q_0 < q$, the initial data satisfying (3.20) do not necessarly satisfy neither (3.14) nor (3.15).

3.3. *Structure of the proof of Theorem 3.1*

Before moving into the details of proof of Theorem 3.1, we explain in this paragraph the outline of this proof.

The general idea of the indirect decay estimate (3.10) lies in the fact that the term v_t, which could be regarded as the viscous damping for the second equaion of (1.1), can be expressed via higher derivatives of u through the weak coupling

$$-\tilde{B}v(t) = u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds.$$

This higher energy decay estmate on the u-equation provides some control over the terms for the energy of the v-equation.

The proof of (3.10) focuses on the case $n = 1$ and it is based on the Lemmas 3.2–3.4 below, where the considered functionals I_1, I_2, J_1, J_2, R_1, and R_2 are inspired from [8,12, 13,18,38–40,42,43] and [10]. The functionals I_2 and J_2 defined in, respectively, Lemmas 3.2 and 3.3 below are used only in case $\tilde{A} \neq A - g_0 B$. Due to the definition of $\mathscr{K}_{i_0}$, all the considered functionals are well defined.

The goal of the proof is to construct the functional F defined in Lemma 3.5 and show that

$$F(t) \geq M_0(E(t) + E_1(t) + \theta_0 E_2(t)), \quad \forall t \in \mathbb{R}_+$$

and

$$G_0(\epsilon_0 E(t)) \leq -C_{11}\frac{G_0(\epsilon_0 E(t))}{E(t)}F'(t) - C_{12}\left(\xi_1 E'(t) + \sum_{i=2}^{5}\xi_i E_i'(t)\right), \quad \forall t \in \mathbb{R}_+, \tag{3.21}$$

where M_0, C_{11}, and C_{12} are positive constants, $\theta_0 = \xi_3 = 0$ if $\tilde{A} = A - g_0 B$, $\theta_0 = \xi_3 = 1$ if $\tilde{A} \neq A - g_0 B$, $(\xi_1, \xi_4) = (1, 0)$ if $j_0 = 0$, $(\xi_1, \xi_4) = (0, 1)$ if $j_0 = 1$, $(\xi_2, \xi_5) = (0, 1)$ if $j_0 = 1$ and $\tilde{A} \neq A - g_0 B$, $(\xi_2, \xi_5) = (1, 0)$ otherwise, E is the energy functional giving by (1.16) and E_i $(i = 2, 3, 4, 5)$ are higher order energy functionals. The estimate (3.21) will follow from the preceding ones proved in Lemmas 3.2–3.4 below, and it demonstrates that a certain function of the energy E is dominated by derivatives of globally bounded functionals. So, integrating the differential inequality (3.21) will ultimately provide the decaying bound (3.10) of E in case $n = 1$. The general case (3.10), for any $n \in \mathbb{N}^*$, is then deduced by induction on n.

3.4. *Proof of Theorem 3.1*

Assume that (A1)–(A8) hold and let $\mathscr{U}_0 \in \mathscr{K}_{i_0}$ and E be the associated energy functional with the solution of (2.8) giving by (1.16). Using (2.8), (2.16) and the fact that

$$E(t) = \frac{1}{2}\|\mathscr{U}(t)\|_{\mathscr{H}}^2, \tag{3.22}$$

we get

$$E'(t) = \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \tag{3.23}$$

Recalling that g is nonincreasing, (3.23) implies that E is nonincreasing, and consequently, (2.8) is dissipative. If $g \equiv 0$, then $E' \equiv 0$; thus, (2.8) is a conservative system. This fact shows that the dissipation resulting from the infinite memory is the unique control considered in the system (1.1).

LEMMA 3.2 *Let us define the functionals*

$$I_1(t) = -\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle$$

and

$$I_2(t) = -\left\langle u_{ttt}(t), \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds\right\rangle - \left\langle \tilde{B}v_t(t), \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds\right\rangle.$$

Then, for any $\epsilon_1,\ \delta_1 > 0$, there exist $c_{\epsilon_1},\ c_{\delta_1} > 0$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} I_1'(t) \le & -(g_0-\epsilon_1)\|u_t(t)\|^2 + \epsilon_1\|A^{\frac{1}{2}}u(t)\|^2 + \epsilon_1\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 \\ & + c_{\epsilon_1}\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} I_2'(t) \le & -(g_0-\delta_1)\|u_{ttt}(t)\|^2 + \delta_1\|A^{\frac{1}{2}}u_{tt}(t)\|^2 + \delta_1\|v_t(t)\|^2 \\ & + c_{\delta_1}\left(\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{ttt}(t,s)\|^2 ds\right). \end{aligned} \tag{3.25}$$

Proof As in [8] and [10], multiplying the first equation of (1.1) by $\int_0^{+\infty} g(s)\eta(t,s)ds$, we get

$$\begin{aligned} 0 = & \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle + \left\langle (A-g_0B)u(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle \\ & + \left\langle \int_0^{+\infty} g(s)B\eta(t,s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle + \left\langle \tilde{B}v(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle. \end{aligned}$$

Using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we obtain

$$\begin{aligned} 0 = & \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds\right\rangle \\ & - g_0\left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}u(t)ds, \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\rangle \\ & + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\rangle \\ & + \left\langle \tilde{B}v(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle. \end{aligned} \tag{3.26}$$

On the other hand, by using the fact that $\partial_t \eta(t,s) = -\partial_s \eta(t,s) + u_t(t)$ (according to (2.13)), we find

$$\begin{aligned}&\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle\\ &\quad = \partial_t \left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle - \left\langle u_t(t), \int_0^{+\infty} g(s)\eta_t(t,s)ds\right\rangle\\ &\quad = -I_1'(t) - g_0\|u_t(t)\|^2 + \left\langle u_t(t), \int_0^{+\infty} g(s)\eta_s(t,s)ds\right\rangle.\end{aligned}$$

Integrating by parts with respect to s in the infinite memory integral, and using the fact that $\lim_{s\to+\infty} g(s)\eta(t,s) = 0$ and $\eta(t,0) = 0$ (according to (A3) and (2.13)), we get

$$\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle = -I_1'(t) - g_0\|u_t(t)\|^2 - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)ds\right\rangle. \tag{3.27}$$

Exploiting (3.26) and (3.27), we deduce

$$\begin{aligned}I_1'(t) &= -g_0\|u_t(t)\|^2\\ &\quad - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)ds\right\rangle + \left\langle \tilde{B}v(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle\\ &\quad + \left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds\right\rangle - g_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\rangle\\ &\quad + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\rangle.\end{aligned}$$

Using Cauchy–Schwarz and Young's inequalities for the last five terms of this equality, and (2.1)–(2.5) to estimate $\|B^{\frac{1}{2}}u(t)\|^2$, $\|\eta(t,s)\|^2$, $\|\tilde{B}v(t)\|^2$, and $-g'(s)$ by $a_0\|A^{\frac{1}{2}}u(t)\|^2$, $\frac{a_0}{a_1}\|A^{\frac{1}{2}}u(t)\|^2$, $\frac{b_1}{\tilde{a}_1}\|\tilde{A}^{\frac{1}{2}}v(t)\|^2$ and $\delta_0 g(s)$, respectively, we get (3.24).

Using the system obtained by differentiating two times the first equation of (1.1) with respect to time t; that is

$$u_{tttt}(t) + Au_{tt}(t) - \int_0^{+\infty} g(s)Bu_{tt}(t-s)ds + \tilde{B}v_{tt}(t) = 0, \quad \forall t > 0, \tag{3.28}$$

multiplying (3.28) by $\int_0^{+\infty} g(s)\eta_{tt}(t,s)ds$, we find as for I_1'

$$\begin{aligned}&I_2'(t) = -g_0\|u_{ttt}(t)\|^2 - \left\langle u_{ttt}(t), \int_0^{+\infty} g'(s)\eta_{tt}(t,s)ds\right\rangle\\ &+ \left\langle A^{\frac{1}{2}}u_{tt}(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta_{tt}(t,s)ds\right\rangle - g_0\left\langle B^{\frac{1}{2}}u_{tt}(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds\right\rangle\\ &- \left\langle \tilde{B}v_t(t), \int_0^{+\infty} g(s)\eta_{ttt}(t,s)ds\right\rangle + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds, \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds\right\rangle.\end{aligned}$$

Then, following the same procedure as before, we get (3.25). □

LEMMA 3.3 *Define the functionals*

$$J_1(t) = \langle u_t(t), u(t)\rangle, \quad J_2(t) = \langle u_{ttt}(t), u_{tt}(t)\rangle + \left\langle \tilde{B}v_t, u_{tt}(t)\right\rangle \text{ and } R_1(t) = \langle v_t(t), v(t)\rangle.$$

Then, for any $\lambda_1,\ \lambda_2,\ \lambda_3,\ \epsilon_2,\ \delta_2 > 0$, *there exist* $c_{\epsilon_2},\ c_{\delta_2} > 0$ *such that*

$$J_1'(t) \le \|u_t(t)\|^2 - (1 - a_0 g_0 - \epsilon_2 - \lambda_1)\|A^{\frac{1}{2}}u(t)\|^2 + \frac{b_1 a_0}{4\lambda_1 a_1 \tilde{a}_1}\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + c_{\epsilon_2}\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.29}$$

$$J_2'(t) \le (1+\lambda_2)\|u_{ttt}(t)\|^2 - (1 - a_0 g_0 - \delta_2)\|A^{\frac{1}{2}}u_{tt}(t)\|^2 + \frac{b_1}{4\lambda_2}\|v_t(t)\|^2 + c_{\delta_2}\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+ \tag{3.30}$$

and

$$R_1'(t) \le \|v_t(t)\|^2 + \frac{a_0 b_1}{4a_1\tilde{a}_1\lambda_3}\|A^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_3)\|\tilde{A}^{\frac{1}{2}}v(t)\|^2, \quad \forall t \in \mathbb{R}_+. \tag{3.31}$$

Proof Multiplying the first equation of (1.1) by $u(t)$, we find

$$0 = \langle u_{tt}(t), u(t)\rangle + \langle (A - g_0 B)u(t), u(t)\rangle + \left\langle \int_0^{+\infty} g(s)B\eta(t,s)ds, u(t)\right\rangle + \left\langle \tilde{B}v(t), u(t)\right\rangle.$$

Consequently, using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we have

$$0 = \partial_t \langle u_t(t), u(t)\rangle - \|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, B^{\frac{1}{2}}u(t)\right\rangle + \left\langle \tilde{B}v(t), u(t)\right\rangle,$$

which implies that

$$J_1'(t) = \|u_t(t)\|^2 - \|A^{\frac{1}{2}}u(t)\|^2 + g_0\|B^{\frac{1}{2}}u(t)\|^2 - \left\langle \tilde{B}v(t), u(t)\right\rangle - \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, B^{\frac{1}{2}}u(t)\right\rangle. \tag{3.32}$$

By using Cauchy–Schwarz and Young's inequalities for the last two terms of (3.32), and (2.1), (2.2), and (2.5) to estimate $\|B^{\frac{1}{2}}u(t)\|^2$, $\|u(t)\|^2$, and $\|\tilde{B}v(t)\|^2$ by $a_0\|A^{\frac{1}{2}}u(t)\|^2$, $\frac{a_0}{a_1}\|A^{\frac{1}{2}}u(t)\|^2$ and $\frac{b_1}{\tilde{a}_1}\|\tilde{A}^{\frac{1}{2}}v(t)\|^2$, respectively, inequality (3.29) holds. Similarly, multiplying the second equation of (1.1) by $v(t)$ and following the same procedure as in the proof of (3.29), we find (3.31). On the other hand, multiplying (3.28) by $u_{tt}(t)$, we find as for J_1'

$$J_2'(t) = \|u_{ttt}(t)\|^2 - \|A^{\frac{1}{2}}u_{tt}(t)\|^2 + g_0\|B^{\frac{1}{2}}u_{tt}(t)\|^2 + \left\langle \tilde{B}v_t(t), u_{ttt}(t)\right\rangle - \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds, B^{\frac{1}{2}}u_{tt}(t)\right\rangle.$$

Then, following the same procedure as in the proof of (3.29), we find (3.30). □

Now, we adapt the approach of [38] to our system (1.1) in objectif to get a crucial estimate.

LEMMA 3.4 *Let R_2 be the functional defined by*

$$R_2(t) = \langle u_{tt}(t), v_t(t)\rangle - \langle u_t(t), v_{tt}(t)\rangle + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle$$

when (3.4) holds and $\tilde{A} = A - g_0 B$,

$$R_2(t) = -\langle u_{tt}(t), v_t(t)\rangle + \langle u_t(t), v_{tt}(t)\rangle - \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle$$

when (3.5) holds and $\tilde{A} = A - g_0 B$,

$$\begin{aligned} R_2(t) = {} & \left\langle \tilde{A}^{-1}Au_{tt}(t), v_t(t)\right\rangle - \left\langle \tilde{A}^{-1}Au_t(t), v_{tt}(t)\right\rangle - g_0\left\langle B^{\frac{1}{2}}u_t(t), B^{\frac{1}{2}}v(t)\right\rangle \\ & + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle \end{aligned}$$

when (3.4) holds and $\tilde{A} \neq A - g_0 B$, and

$$\begin{aligned} R_2(t) = {} & -\left\langle \tilde{A}^{-1}Au_{tt}(t), v_t(t)\right\rangle + \left\langle \tilde{A}^{-1}Au_t(t), v_{tt}(t)\right\rangle \\ & + g_0\left\langle B^{\frac{1}{2}}u_t(t), B^{\frac{1}{2}}v(t)\right\rangle - \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle \end{aligned}$$

when (3.5) holds and $\tilde{A} \neq A - g_0 B$. Then, for any $\delta_3, \epsilon_3, \epsilon_4 > 0$, there exists $c_{\epsilon_3}, c_{\epsilon_4} > 0$ such that, for all $t \in \mathbb{R}_+$,

$$R_2'(t) \leq -b_0\|v_t(t)\|^2 + \sqrt{b_1}\|u_t(t)\|^2 + \epsilon_3\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + c_{\epsilon_3}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds \tag{3.33}$$

in case $\tilde{A} = A - g_0 B$, and

$$\begin{aligned} R_2'(t) \leq {} & -(b_0 - \epsilon_3)\|v_t(t)\|^2 + \sqrt{b_1 d_0}\|u_t(t)\|^2 + \frac{d_0+1}{2\epsilon_3}\|u_{ttt}(t)\|^2 + \frac{g_0^2 a_0^2 \tilde{a}_2}{4\delta_3}\|A^{\frac{1}{2}}u_{tt}(t)\|^2 \\ & + (\delta_3 + \epsilon_4)\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + c_{\epsilon_4}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds \end{aligned} \tag{3.34}$$

in case $\tilde{A} \neq A - g_0 B$, where d_0 is a positive constant defined by

$$\|\tilde{A}^{-1}Aw\|^2 \leq d_0\|w\|^2, \quad \forall w \in D(A) \tag{3.35}$$

(since $\tilde{A}^{-1}A$ is bounded thanks to (3.2)).

Proof
Case 1 $\tilde{A} = A - g_0 B$: considering the equations obtained by differentiating the equations of (1.1) with respect to time t; that is

$$u_{ttt}(t) + Au_t(t) - \int_0^{+\infty} g(s)Bu_t(t-s)ds + \tilde{B}v_t(t) = 0, \quad \forall t > 0 \tag{3.36}$$

and

$$v_{ttt}(t) + \tilde{A} v_t(t) + \tilde{B} u_t(t) = 0, \quad \forall t > 0, \tag{3.37}$$

and multiplying (3.36) and (3.37) by $v_t(t)$ and $u_t(t)$, respectively, we get

$$\begin{aligned} 0 &= \langle u_{ttt}(t), v_t(t)\rangle + \langle (A - g_0 B)u_t(t), v_t(t)\rangle + \left\langle \int_0^{+\infty} g(s) B\eta_t(t,s)ds, v_t(t)\right\rangle \\ &\quad + \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle - \langle v_{ttt}(t), u_t(t)\rangle - \left\langle \tilde{A} v_t(t), u_t(t)\right\rangle - \left\langle \tilde{B} u_t(t), u_t(t)\right\rangle \\ &= \partial_t \left(\langle u_{tt}(t), v_t(t)\rangle - \langle u_t(t), v_{tt}(t)\rangle + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle \right) \\ &\quad - \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle - \left\langle \tilde{B} u_t(t), u_t(t)\right\rangle + \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle, \end{aligned}$$

since $\langle (A - g_0 B)u_t(t), v_t(t)\rangle - \left\langle \tilde{A} v_t(t), u_t(t)\right\rangle = 0$ (because $\tilde{A} = A - g_0 B$ and $\tilde{A}$ is self-adjoint). Therefore,

$$\begin{aligned} &\partial_t \left(\langle u_{tt}(t), v_t(t)\rangle - \langle u_t(t), v_{tt}(t)\rangle + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle \right) \\ &\quad = \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle + \left\langle \tilde{B} u_t(t), u_t(t)\right\rangle - \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle. \end{aligned} \tag{3.38}$$

Consequently, using Cauchy–Schwarz and Young's inequalities for the first term of the right-hand side of (3.38), and (2.1) and (3.2) to estimate $\|B^{\frac{1}{2}} v(t)\|^2$ by $a_0 \tilde{a}_2 \|\tilde{A}^{\frac{1}{2}} v(t)\|^2$, and using (2.5) and (3.4) to estimate the last two terms of the right-hand side of (3.38), we get (3.33) when $\tilde{A} = A - g_0 B$ and (3.4) holds. Similarly, multiplying (3.38) by -1 and following the same procedure, we find (3.33) when $\tilde{A} = A - g_0 B$ and (3.5) holds.

Case 2 $\tilde{A} \neq A - g_0 B$: multiplying (3.36) and (3.37) by $v_t(t)$ and $\tilde{A}^{-1} A u_t(t)$, respectively, we get

$$\begin{aligned} 0 &= \langle u_{ttt}(t), v_t(t)\rangle + \langle (A - g_0 B)u_t(t), v_t(t)\rangle + \left\langle \int_0^{+\infty} g(s) B\eta_t(t,s)ds, v_t(t)\right\rangle \\ &\quad + \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle - \left\langle v_{ttt}(t), \tilde{A}^{-1} A u_t(t)\right\rangle - \left\langle \tilde{A} v_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle - \left\langle \tilde{B} u_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle \\ &= \partial_t \left(\left\langle \tilde{A}^{-1} A u_{tt}(t), v_t(t)\right\rangle - \left\langle \tilde{A}^{-1} A u_t(t), v_{tt}(t)\right\rangle - g_0 \left\langle B^{\frac{1}{2}} u_t(t), B^{\frac{1}{2}} v(t)\right\rangle \right. \\ &\quad \left. + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle \right) - \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle \\ &\quad - \left\langle \tilde{B} u_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle + g_0 \left\langle B^{\frac{1}{2}} u_{tt}(t), B^{\frac{1}{2}} v(t)\right\rangle + \langle u_{ttt}(t), v_t(t)\rangle \\ &\quad - \left\langle \tilde{A}^{-1} A u_{ttt}(t), v_t(t)\right\rangle + \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle, \end{aligned}$$

since $\langle A u_t(t), v_t(t)\rangle - \left\langle \tilde{A}^{-1} A u_t(t), \tilde{A} v_t(t)\right\rangle = 0$ (because $\tilde{A}$ is self-adjoint). Therefore

$$\begin{aligned} &\partial_t \left(\left\langle \tilde{A}^{-1} A u_{tt}(t), v_t(t)\right\rangle - \left\langle \tilde{A}^{-1} A u_t(t), v_{tt}(t)\right\rangle - g_0 \left\langle B^{\frac{1}{2}} u_t(t), B^{\frac{1}{2}} v(t)\right\rangle \right. \\ &\quad \left. + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s)ds, B^{\frac{1}{2}} v(t)\right\rangle \right) \end{aligned}$$

$$= \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s) ds, \, B^{\frac{1}{2}} v(t) \right\rangle + \left\langle \tilde{B} u_t(t), \, \tilde{A}^{-1} A u_t(t) \right\rangle - g_0 \left\langle B^{\frac{1}{2}} u_{tt}(t), \, B^{\frac{1}{2}} v(t) \right\rangle$$
$$- \langle u_{ttt}(t), v_t(t) \rangle + \left\langle \tilde{A}^{-1} A u_{ttt}(t), v_t(t) \right\rangle - \left\langle \tilde{B} v_t(t), v_t(t) \right\rangle. \tag{3.39}$$

Consequently, using Cauchy-Schwarz and Young's inequalities for the first four terms of the right-hand side of (3.39), applying (3.2), (2.5) and (2.1) to estimate $\|B^{\frac{1}{2}} v(t)\|^2$, $\|\tilde{B} u_t(t)\|$ and $\|B^{\frac{1}{2}} u_{tt}(t)\|^2$ by $a_0 \tilde{a}_2 \|\tilde{A}^{\frac{1}{2}} v(t)\|^2$, $\sqrt{b_1}\|u_t(t)\|$ and $a_0 \|A^{\frac{1}{2}} u_{tt}(t)\|^2$, respectively, using (3.35) and (3.4) to estimate the last two terms of the right-hand side of (3.39), we get (3.34) when $\tilde{A} \neq A - g_0 B$ and (3.4) holds. Similarly, multiplying (3.39) by -1 and following the same procedure, we find (3.34) when $\tilde{A} \neq A - g_0 B$ and (3.5) holds. □

Before proving the next lemma, let us consider, for $k = 1, 2, 3$,

$$E_k(t) = \frac{1}{2} \|\partial_t^k \mathscr{U}(t)\|_{\mathscr{H}}^2, \quad E_4(t) = \frac{1}{2} \|A^{\frac{1}{2}} \mathscr{U}(t)\|_{\mathscr{H}}^2 \quad \text{and} \quad E_5(t) = \frac{1}{2} \|A^{\frac{1}{2}} \partial_t^2 \mathscr{U}(t)\|_{\mathscr{H}}^2. \tag{3.40}$$

Similarly to (3.23), we have

$$E_k'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \|B^{\frac{1}{2}} \partial_t^k \eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.41}$$

$$E_4'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \|A^{\frac{1}{2}} B^{\frac{1}{2}} \eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+ \tag{3.42}$$

and

$$E_5'(t) = \frac{1}{2} \int_0^{+\infty} g'(s) \|A^{\frac{1}{2}} B^{\frac{1}{2}} \eta_{tt}(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \tag{3.43}$$

LEMMA 3.5 *There exist positive constants N_i, M_i $(i = 0, 1, 2)$ and C_i $(i = 0, 1)$ such that the functional*

$$F(t) = N_0(E(t) + E_1(t)) + N_1 I_1(t) + M_1 J_1(t) + C_1 R_1(t) + R_2(t) \tag{3.44}$$

if $\tilde{A} = A - g_0 B$, and

$$F(t) = N_0(E(t) + E_1(t) + E_2(t)) + N_1 I_1(t) + N_2 I_2(t)$$
$$+ M_1 J_1(t) + M_2 J_2(t) + C_1 R_1(t) + R_2(t) \tag{3.45}$$

if $\tilde{A} \neq A - g_0 B$, satisfies, for all $t \in \mathbb{R}_+$,

$$F(t) \geq M_0(E(t) + E_1(t)) \tag{3.46}$$

and

$$F'(t) \leq -M_0 E(t) + C_0 \int_0^{+\infty} g(s) \left(\|A^{\frac{1}{2}} \eta(t,s)\|^2 + \|B^{\frac{1}{2}} \eta_{tt}(t,s)\|^2 \right) ds \tag{3.47}$$

if $\tilde{A} = A - g_0 B$, and

$$F(t) \geq M_0(E(t) + E_1(t) + E_2(t)) \tag{3.48}$$

and

$$F'(t) \le -M_0 E(t) + C_0 \int_0^{+\infty} g(s) \left(\|A^{\frac{1}{2}}\eta(t,s)\|^2 + \|A^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 + \|B^{\frac{1}{2}}\eta_{ttt}(t,s)\|^2 \right) ds \tag{3.49}$$

if $\tilde{A} \neq A - g_0 B$.

Proof First, we prove (3.46) and (3.48). Using (2.12) and the fact that $\tilde{B}$ is self-adjoint, we find, for all $t \in \mathbb{R}_+$,

$$E(t) \ge \frac{c_0}{2} \left(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \right). \tag{3.50}$$

Similarly,

$$E_1(t) \ge \frac{c_0}{2} \left(\|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 + \|A^{\frac{1}{2}}u_t(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v_t(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_t(t,s)\|^2 ds \right) \tag{3.51}$$

and

$$E_2(t) \ge \frac{c_0}{2} \left(\|u_{ttt}(t)\|^2 + \|v_{ttt}(t)\|^2 + \|A^{\frac{1}{2}}u_{tt}(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v_{tt}(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds \right). \tag{3.52}$$

From (3.50)–(3.52) and the definition of I_i, J_i and R_i $(i = 1, 2)$, we see that there exists a positive constant L (not depending on N_0) such that

$$F(t) \ge (N_0 - L)(E(t) + E_1(t)), \quad \forall t \in \mathbb{R}_+$$

in case $\tilde{A} = A - g_0 B$, and

$$F(t) \ge (N_0 - L)(E(t) + E_1(t) + E_2(t)), \quad \forall t \in \mathbb{R}_+$$

in case $\tilde{A} \neq A - g_0 B$. Thus, choosing $N_0 > L$, (3.46) and (3.48) hold, for any

$$M_0 \le N_0 - L. \tag{3.53}$$

Second, we prove (3.47) and (3.49) by distinguishing two cases.

Case 1 $\tilde{A} = A - g_0 B$: by combining (3.24), (3.29), (3.31), and (3.33), and noting that $E',\ E'_k \le 0$ (according to (3.23) and (3.41)), we get

$$\begin{aligned} F'(t) \le & -L_1\|u_t(t)\|^2 - L_2\|v_t(t)\|^2 - L_3\|A^{\frac{1}{2}}u(t)\|^2 - L_4\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 \\ & + (N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2}) \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ & + c_{\epsilon_3} \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{3.54}$$

where

$$\begin{cases} L_1 = g_0 N_1 - M_1 - \sqrt{b_1} - \epsilon_1 N_1, \\ L_2 = b_0 - C_1, \\ L_3 = (1 - a_0 g_0 - \lambda_1) M_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 - \epsilon_1 N_1 - \epsilon_2 M_1, \\ L_4 = (1 - \lambda_3) C_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 - \epsilon_1 N_1 - \epsilon_3. \end{cases}$$

We choose $\lambda_1 = \frac{1}{2}(1 - a_0 g_0)$ (which is positive since (2.3)), $\lambda_3 = \frac{1}{2}$, $0 < C_1 < b_0$ and

$$N_1 > \frac{a_0 b_1 C_1}{g_0 a_1 \tilde{a}_1 (1 - a_0 g_0)} + \frac{\sqrt{b_1}}{g_0}$$

(note that because $g(0) > 0$ according to (A8), then $g_0 > 0$). Later we take

$$\frac{a_0 b_1 C_1}{a_1 \tilde{a}_1 (1 - a_0 g_0)} < M_1 < \min\left\{ g_0 N_1 - \sqrt{b_1}, \frac{a_1 \tilde{a}_1 (1 - a_0 g_0) C_1}{a_0 b_1} \right\}$$

(M_1 exists due to (3.3) and the definition of N_1). These choices imply that

$$g_0 N_1 - M_1 - \sqrt{b_1} > 0, \quad b_0 - C_1 > 0, \quad (1 - a_0 g_0 - \lambda_1) M_1 - \frac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 > 0$$

and

$$(1 - \lambda_3) C_1 - \frac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 > 0.$$

Finally, we choose $\epsilon_3 = \epsilon_2 = \epsilon_1$ and ϵ_1 small enough such that $L_i > 0$, $i = 1, \cdots, 4$. On the other hand, using Young inequality, (2.1), (2.2), and (2.5), we find

$$\begin{aligned} \left\langle \tilde{B} u(t), v(t) \right\rangle &\le \frac{1}{2} (\|\tilde{B} u(t)\|^2 + \|v(t)\|^2) \\ &\le \frac{1}{2} \left(\frac{b_1 a_0}{a_1} \|A^{\frac{1}{2}} u(t)\|^2 + \frac{1}{\tilde{a}_1} \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 \right), \quad \forall t \in \mathbb{R}_+, \end{aligned}$$

therefore, from (1.16),

$$\begin{aligned} E(t) \le C_2 \Bigg(&\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 + \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 \\ &+ \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds \Bigg), \end{aligned} \tag{3.55}$$

where

$$C_2 = \frac{1}{2} \max\left\{ 1 + \frac{b_1 a_0}{a_1}, 1 + \frac{1}{\tilde{a}_1} \right\}.$$

By combining (3.54) and (3.55), (3.47) holds, for any

$$M_0 \le \frac{1}{C_2} \min\{L_1, L_2, L_3, L_4\} \tag{3.56}$$

and (using (2.1) to estimate $\|B^{\frac{1}{2}} \eta(t,s)\|^2$ by $a_0 \|A^{\frac{1}{2}} \eta(t,s)\|^2$)

$$C_0 = \max\left\{ c_{\epsilon_3}, N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2} + a_0 \min\{L_1, L_2, L_3, L_4\} \right\}.$$

So, (3.46) and (3.47) hold, for any $M_0 > 0$ satisfying (3.53) and (3.56).

Case 2 $\tilde{A} \neq A - g_0 B$: similarly to the proof in case $\tilde{A} = A - g_0 B$, by combining (3.24), (3.25), (3.29)-(3.31) and (3.34), and noting that $E', \ E'_k \leq 0$ (according to (3.23) and (3.41)), we get (using also (2.1) to estimate $\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2$ by $a_0\|A^{\frac{1}{2}}\eta_{tt}(t,s)\|^2$)

$$\begin{aligned} F'(t) \leq & -L_1\|u_t(t)\|^2 - L_2\|v_t(t)\|^2 - L_3\|A^{\frac{1}{2}}u(t)\|^2 - L_4\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 - L_5\|u_{ttt}(t)\|^2 \\ & - L_6\|A^{\frac{1}{2}}u_{tt}(t)\|^2 + (N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2}) \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ & + (N_2 c_{\delta_1} + M_2 c_{\delta_2} + a_0 c_{\epsilon_4}) \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds \\ & + N_2 c_{\delta_1} \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{ttt}(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{3.57}$$

where

$$\begin{cases} L_1 = g_0 N_1 - M_1 - \sqrt{b_1 d_0} - \epsilon_1 N_1, \\ L_2 = b_0 - C_1 - \epsilon_3 - \dfrac{b_1 M_2}{4\lambda_2} - \delta_1 N_2, \\ L_3 = (1 - a_0 g_0 - \lambda_1) M_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 - \epsilon_1 N_1 - \epsilon_2 M_1, \\ L_4 = (1 - \lambda_3) C_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 - \delta_3 - \epsilon_1 N_1 - \epsilon_4, \\ L_5 = g_0 N_2 - (1 + \lambda_2) M_2 - \dfrac{d_0 + 1}{2\epsilon_3} - \delta_1 N_2, \\ L_6 = (1 - a_0 g_0) M_2 - \dfrac{a_0^2 g_0^2 \tilde{a}_2}{4\delta_3} - \delta_1 N_2 - \delta_2 M_2. \end{cases}$$

We choose

$$\lambda_1 = \frac{1}{2}(1 - a_0 g_0), \quad \lambda_3 = \frac{1}{2} \quad \text{and} \quad 0 < \delta_3 < \frac{b_0\left((a_1\tilde{a}_1)^2(1 - a_0 g_0)^2 - (a_0 b_1)^2\right)}{2(a_1\tilde{a}_1)^2(1 - a_0 g_0)^2}$$

(δ_3 exists thanks to (3.3)). Next, we take

$$\frac{2\delta_3(a_1\tilde{a}_1)^2(1 - a_0 g_0)^2}{(a_1\tilde{a}_1)^2(1 - a_0 g_0)^2 - (a_0 b_1)^2} < C_1 < b_0 \quad \text{and} \quad M_2 > \frac{a_0^2 g_0^2 \tilde{a}_2}{4(1 - a_0 g_0)\delta_3}.$$

Later we pick

$$0 < \epsilon_3 < b_0 - C_1, \qquad \frac{a_0 b_1 C_1}{a_1\tilde{a}_1(1 - a_0 g_0)} < M_1 < \frac{a_1\tilde{a}_1(1 - a_0 g_0)(C_1 - 2\delta_3)}{a_0 b_1}$$

and

$$\lambda_2 > \frac{b_1 M_2}{4(b_0 - C_1 - \epsilon_3)}.$$

Finally, we choose

$$N_1 > \frac{\sqrt{b_1 d_0} + M_1}{g_0} \quad \text{and} \quad N_2 > \frac{1}{g_0}\left(\frac{d_0 + 1}{2\epsilon_3} + (1 + \lambda_2) M_2\right).$$

These choices imply that

$$g_0 N_1 - M_1 - \sqrt{b_1 d_0} > 0, \quad b_0 - C_1 - \epsilon_3 - \frac{b_1 M_2}{4\lambda_2} > 0,$$

$$(1-a_0g_0-\lambda_1)M_1-\frac{a_0b_1}{4a_1\tilde{a}_1\lambda_3}C_1>0,\quad (1-\lambda_3)C_1-\frac{a_0b_1}{4a_1\tilde{a}_1\lambda_1}M_1-\delta_3>0,$$

$$g_0N_2-(1+\lambda_2)M_2-\frac{d_0+1}{2\epsilon_3}>0\quad\text{and}\quad (1-a_0g_0)M_2-\frac{a_0^2g_0^2\tilde{a}_2}{4\delta_3}>0.$$

At the end, we take $\epsilon_4=\epsilon_2=\delta_2=\delta_1=\epsilon_1$ and ϵ_1 small enough such that $L_i>0$, $i=1,\cdots,6$. By combining (3.55) and (3.57), we find (3.49), for any M_0 satisfying (3.56) and

$$C_0=\max\left\{N_1c_{\epsilon_1}+M_1c_{\epsilon_2}+a_0\min\{L_1,L_2,L_3,L_4,L_5,L_6\},N_2c_{\delta_1}\right.$$
$$\left.+M_2c_{\delta_2}+a_0c_{\epsilon_4},N_2c_{\delta_1}\right\}.$$

So, (3.48) and (3.49) hold, for any $M_0>0$ satisfying (3.53) and (3.56). □

Now, we estimate the integral terms in (3.47) and (3.49). Under the condition (3.6) and using (3.23), we have

$$\int_0^{+\infty}g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2ds\le\frac{-2}{\delta}E'(t),\quad\forall t\in\mathbb{R}_+.\tag{3.58}$$

In case where (3.7) holds and (3.6) does not hold, we prove this lemma.

Lemma 3.6 *There exists a positive constant C_3 such that, for any $\epsilon_0>0$, the following inequality holds:*

$$G'(\epsilon_0E(t))\int_0^{+\infty}g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2ds\le -C_3E'(t)+C_3\epsilon_0E(t)G'(\epsilon_0E(t)),\quad\forall t\in\mathbb{R}_+.\tag{3.59}$$

Proof First, we note that (3.58) and (3.59) are equivalent when $G=Id$; so (3.59) generalizes (3.58) (which represents the kernels g converging exponentially to zero at infinity) to the class of kernels g having a decay rate smaller than the exponential one. We have proved estimate (3.59), first, in [8] under condition (3.15), and then applied it in [12] and [13]. The proof of (3.59), for initial data $\mathscr{U}_0\in\mathscr{K}_{i_0}$ and under the condition (3.7), is slightly different.

We note that, if $g'(s_0)=0$, for some $s_0\ge 0$, then $g(s_0)=0$ because $G^{-1}(0)=0$ and $s\mapsto\frac{g(s)}{G^{-1}(-g'(s))}$ is bounded (thanks to (3.7)), and therefore, $g(s)=0$, for all $s\ge s_0$, because g is nonnegative and nonincreasing. This implies that the infinite integrals in (3.47) and (3.49) are effective only on $[0,s_0]$. In the same time, if $E(s_0)=0$, for some $s_0\ge 0$, then $E(s)=0$, for all $s\ge s_0$, since E is nonnegative and nonincreasing, consequently, (3.10) is satisfied, for any $c_n\ge s_0G_n^{-1}(E(0))$. Thus, without loss of generality, we can assume that $g'<0$ and $E>0$ on $\mathbb{R}_+$.

Let $t\in\mathbb{R}_+$. Because E is nonincreasing and using (2.1), (3.50) implies that

$$\|B^{\frac{1}{2}}\eta(t,s)\|^2\le 2\left(\|B^{\frac{1}{2}}u(t)\|^2+\|B^{\frac{1}{2}}u(t-s)\|^2\right)$$
$$\le\frac{4a_0}{c_0}E(0)+2\|B^{\frac{1}{2}}u(t-s)\|^2,\quad\forall s\in\mathbb{R}_+.$$

Then, for

$$M(t,s) := \begin{cases} \dfrac{8a_0}{c_0} E(0) & \text{if } 0 \le s \le t, \\ \dfrac{4a_0}{c_0} E(0) + 2\|B^{\frac{1}{2}} u_0(s-t)\|^2 & \text{if } s > t \ge 0, \end{cases} \tag{3.60}$$

we conclude that

$$\|B^{\frac{1}{2}} \eta(t,s)\|^2 \le M(t,s), \quad \forall t,\ s \in \mathbb{R}_+. \tag{3.61}$$

Let $\tau_1(t,s),\ \tau_2(t,s) > 0$ (which will be fixed later on), $\epsilon_0 > 0$ and $K(s) = \frac{s}{G^{-1}(s)}$, for $s \in \mathbb{R}_+$ ($K(0) = 0$ because $\lim_{s\to 0^+} \dfrac{s}{G^{-1}(s)} = \lim_{\tau \to 0^+} \dfrac{G(\tau)}{\tau} = G'(0) = 0$). The function K is nondecreasing. Indeed, the fact that G^{-1} is concave and $G^{-1}(0) = 0$ implies that, for any $0 \le s_1 < s_2$,

$$K(s_1) = \frac{s_1}{G^{-1}\left(\frac{s_1}{s_2} s_2 + \left(1 - \frac{s_1}{s_2}\right) 0\right)} \le \frac{s_1}{\frac{s_1}{s_2} G^{-1}(s_2) + \left(1 - \frac{s_1}{s_2}\right) G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2).$$

Then, using (3.61),

$$K\left(-\tau_2(t,s) g'(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2\right) \le K\left(-M(t,s)\tau_2(t,s) g'(s)\right), \quad \forall s \in \mathbb{R}_+.$$

Using this inequality, we arrive at

$$\begin{aligned} &\int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds \\ &\quad = \frac{1}{G'(\epsilon_0 E(t))} \int_0^{+\infty} \frac{1}{\tau_1(t,s)} G^{-1}\left(-\tau_2(t,s) g'(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2\right) \\ &\qquad \times \frac{\tau_1(t,s) G'(\epsilon_0 E(t)) g(s)}{-\tau_2(t,s) g'(s)} K\left(-\tau_2(t,s) g'(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2\right) ds \\ &\quad \le \frac{1}{G'(\epsilon_0 E(t))} \int_0^{+\infty} \frac{1}{\tau_1(t,s)} G^{-1}\left(-\tau_2(t,s) g'(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2\right) \\ &\qquad \times \frac{\tau_1(t,s) G'(\epsilon_0 E(t)) g(s)}{-\tau_2(t,s) g'(s)} K\left(-M(t,s)\tau_2(t,s) g'(s)\right) ds \\ &\quad \le \frac{1}{G'(\epsilon_0 E(t))} \int_0^{+\infty} \frac{1}{\tau_1(t,s)} G^{-1}\left(-\tau_2(t,s) g'(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2\right) \\ &\qquad \times \frac{M(t,s)\tau_1(t,s) G'(\epsilon_0 E(t)) g(s)}{G^{-1}(-M(t,s)\tau_2(t,s) g'(s))} ds. \end{aligned}$$

Let $G^*(s) = \sup_{\tau \in \mathbb{R}_+} \{s\tau - G(\tau)\}$, for $\tau \in \mathbb{R}_+$, denote the dual function of G. Thanks to (A8), G' is increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, and then, for any $s \in \mathbb{R}_+$, the function $\tau \mapsto s\tau - G(\tau)$ reachs its maximum on $\mathbb{R}_+$ at the unique point $\tau = (G')^{-1}(s)$. Therfore,

$$G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)), \quad \forall s \in \mathbb{R}_+.$$

Using Young's inequality: $s_1 s_2 \le G(s_1) + G^*(s_2)$, for

$$s_1 = G^{-1}\left(-\tau_2(t,s) g'(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2\right) \text{ and } s_2 = \frac{M(t,s)\tau_1(t,s) G'(\epsilon_0 E(t)) g(s)}{G^{-1}(-M(t,s)\tau_2(t,s) g'(s))},$$

we get

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \leq \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{-\tau_2(t,s)}{\tau_1(t,s)} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{1}{\tau_1(t,s)} G^*\left(\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\right) ds.$$

Using the fact that $G^*(s) \leq s(G')^{-1}(s)$, we get

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \leq \frac{-1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{\tau_2(t,s)}{\tau_1(t,s)} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + \int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))} (G')^{-1}\left(\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\right) ds.$$

Thanks to (3.7), $\sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} := m_1 < +\infty$. Then, using the fact that $(G')^{-1}$ is nondecreasing and choosing $\tau_2(t,s) = \frac{1}{M(t,s)}$, we get

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \leq \frac{-1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{1}{M(t,s)\tau_1(t,s)} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + \int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-g'(s))} (G')^{-1}\left(m_1 M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))\right) ds.$$

Choosing $\tau_1(t,s) = \frac{1}{m_1 M(t,s)}$ and using (3.23) and the fact that

$$\sup_{t\in\mathbb{R}_+} \int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-g'(s))} ds =: m_2 < +\infty$$

(due to (3.7) and (3.9)), we obtain

$$\begin{aligned}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds &\leq \frac{-m_1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ &\quad + \epsilon_0 E(t)\int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-g'(s))} ds \\ &\leq \frac{-2m_1}{G'(\epsilon_0 E(t))} E'(t) + \epsilon_0 m_2 E(t),\end{aligned}$$

which gives (3.59) with $C_3 = \max\{2m_1, m_2\}$. □

Now, we go back to (3.47) and (3.49). Similarly to (3.58) and (3.59), and using (3.41)–(3.43), we find, for $k = 1, 2, 3$,

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\partial_t^k \eta(t,s)\|^2 ds \leq \frac{-2}{\delta} E_k'(t), \quad \forall t \in \mathbb{R}_+, \tag{3.62}$$

$$\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta(t,s)\|^2 ds \leq \frac{-2}{\delta}E_4'(t), \quad \forall t \in \mathbb{R}_+ \tag{3.63}$$

and

$$\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds \leq \frac{-2}{\delta}E_5'(t), \quad \forall t \in \mathbb{R}_+ \tag{3.64}$$

if (3.6) holds. Otherwise, when (3.7) holds and (3.6) does not hold, we get, for any $\epsilon_0 > 0$ and $t \in \mathbb{R}_+$,

$$G'(\epsilon_0 E(t)) \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\partial_t^k \eta(t,s)\|^2 ds \leq -C_{3+k}E_k'(t) + C_{3+k}\epsilon_0 E(t)G'(\epsilon_0 E(t)), \tag{3.65}$$

$$G'(\epsilon_0 E(t)) \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta(t,s)\|^2 ds \leq -C_7 E_4'(t) + C_7\epsilon_0 E(t)G'(\epsilon_0 E(t)) \tag{3.66}$$

and

$$G'(\epsilon_0 E(t)) \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds \leq -C_8 E_5'(t) + C_8\epsilon_0 E(t)G'(\epsilon_0 E(t)) \tag{3.67}$$

where C_4, C_5, C_6, C_7, and C_8 are defined as C_3 with, respectively, $\|B^{\frac{1}{2}}\partial_s u_0(s-t)\|^2$, $\|B^{\frac{1}{2}}\partial_s^2 u_0(s-t)\|^2$, $\|B^{\frac{1}{2}}\partial_s^3 u_0(s-t)\|^2$, $\|A^{\frac{1}{2}}B^{\frac{1}{2}}u_0(s-t)\|^2$, and $\|A^{\frac{1}{2}}B^{\frac{1}{2}}\partial_s^2 u_0(s-t)\|^2$ instead of $\|B^{\frac{1}{2}}u_0(s-t)\|^2$, and $E_1(0)$, $E_2(0)$, $E_3(0)$, $E_4(0)$ and $E_5(0)$ instead of $E(0)$ in the definition (3.60) of $M(t,s)$. Therefore, using (3.1) to estimate $\|A^{\frac{1}{2}}\partial_s^k\eta\|^2$ by $a_2\|A^{\frac{j_0}{2}}B^{\frac{1}{2}}\partial_s^k\eta\|^2$, for $k = 0, 2$, (3.47) and (3.49), we get, for some positive constants C_9 and C_{10} (do not depending on ϵ_0),

$$F'(t) \leq -M_0 E(t) - C_9(\xi_1 E'(t) + \xi_2 E_2'(t) + \xi_3 E_3'(t) + \xi_4 E_4'(t) + \xi_5 E_5'(t)), \quad \forall t \in \mathbb{R}_+$$

if (3.6) holds, and

$$\begin{aligned} G'(\epsilon_0 E(t))F'(t) \leq& -(M_0 - C_{10}\epsilon_0)E(t)G'(\epsilon_0 E(t)) \\ &- C_{10}(\xi_1 E'(t) + \xi_2 E_2'(t) + \xi_3 E_3'(t) + \xi_4 E_4'(t) + \xi_5 E_5'(t)), \ \forall t \in \mathbb{R}_+ \end{aligned}$$

if (3.7) holds and (3.6) does not hold, where $(\xi_1, \xi_4) = (1, 0)$ if $j_0 = 0$, $(\xi_1, \xi_4) = (0, 1)$ if $j_0 = 1$, $(\xi_2, \xi_5) = (0, 1)$ if $j_0 = 1$ and $\tilde{A} \neq A - g_0 B$, $(\xi_2, \xi_5) = (1, 0)$ otherwise, $\xi_3 = 0$ if $\tilde{A} = A - g_0 B$, and $\xi_3 = 1$ if $\tilde{A} \neq A - g_0 B$. By choosing $0 < \epsilon_0 < \frac{M_0}{C_{10}}$, we see that, for some positive constants C_{11} and C_{12}, and for all $t \in \mathbb{R}_+$,

$$\begin{aligned} G_0(\epsilon_0 E(t)) \leq& -C_{11}\frac{G_0(\epsilon_0 E(t))}{E(t)}F'(t) \\ &- C_{12}(\xi_1 E'(t) + \xi_2 E_2'(t) + \xi_3 E_3'(t) + \xi_4 E_4'(t) + \xi_5 E_5'(t)), \end{aligned} \tag{3.68}$$

where G_0 is defined in (3.11). By integrating (3.69) over $[0, T]$, for $T > 0$, and using the fact that $F,\ E,\ E_2,\ E_3,\ E_4,\ E_5 > 0$ (due to (3.46) and (3.48)), $G_0(\epsilon_0 E)$ and $\frac{G_0(\epsilon_0 E)}{E}$ are nonincreasing, we find

$$\begin{aligned} G_0(\epsilon_0 E(T))T &\leq \int_0^T G_0(\epsilon_0 E(t))dt \\ &\leq c_1, \end{aligned}$$

where

$$c_1 = \max\left\{\frac{1}{\epsilon_0}, C_{11}\frac{G_0(\epsilon_0 E(0))}{E(0)}F(0) + C_{12}\left(\xi_1 E(0) + \xi_2 E_2(0)\right.\right.$$
$$\left.\left. + \xi_3 E_3(0) + \xi_4 E_4(0) + \xi_5 E_5(0)\right)\right\}.$$

This proves (3.10) for $n = 1$.

Now, suppose that (3.10) holds and let $\mathscr{U}_0 \in \mathscr{K}_{i_0(n+1)}$. We have $A^{\frac{i_0}{2}}\mathscr{U}(0) \in \mathscr{K}_{i_0 n}$, $\partial_t^k \mathscr{U}(0) \in \mathscr{K}_{i_0 n}$, for $k = 1, 2$ if $i_0 = 2$, and $k = 1, 2, 3$ if $i_0 = 3$, and $A^{\frac{i_0}{2}}\partial_t^2\mathscr{U}(0) \in \mathscr{K}_{3n}$ (thanks to Theorem 2.2 and the definition of $\mathscr{K}_n$), and then (3.10) implies that, for $k = 1, \cdots, 5$,

$$E(t) \le c_n G_n\left(\frac{c_n}{t}\right) \quad \text{and} \quad E_k(t) \le \theta_n^k G_n\left(\frac{\theta_n^k}{t}\right), \tag{3.69}$$

where θ_n^k is a positive constant. On the other hand, for some positive consatant C_{13} (according to the definition of F, E, E_i, I_i, J_i and R_i),

$$F(t) \le C_{13}(E(t) + E_1(t)), \quad \forall t \in \mathbb{R}_+ \tag{3.70}$$

if $\tilde{A} = A - g_0 B$, and

$$F(t) \le C_{13}(E(t) + E_1(t) + E_2(t)), \quad \forall t \in \mathbb{R}_+ \tag{3.71}$$

if $\tilde{A} \ne A - g_0 B$. Integrating (3.69) over $[T, 2T]$, for $T > 0$, and using (3.70), (3.71) and the fact that $G_0(\epsilon_0 E)$ and $\dfrac{G_0(\epsilon_0 E)}{E}$ are nonincreasing, we deduce that, for for all $T > 0$,

$$\begin{aligned} G_0(\epsilon_0 E(2T))T &\le \int_T^{2T} G_0(\epsilon_0 E(t))dt \\ &\le C_{14}(E(T) + E_1(T) + E_2(T) + \xi_3 E_3(T) + \xi_4 E_4(T) + \xi_5 E_5(T)), \end{aligned} \tag{3.72}$$

where

$$C_{14} = \max\left\{C_{12}, C_{11}C_{13}\frac{G_0(\epsilon_0 E(0))}{E(0)}\right\}.$$

From (3.69) and (3.72), we get, for all $T > 0$,

$$\begin{aligned} E(2T) \le \frac{1}{\epsilon_0}G_0^{-1}\Bigg(\frac{2C_{14}}{2T}\Bigg(& c_n G_n\left(\frac{2c_n}{2T}\right) + \theta_n^1 G_n\left(\frac{2\theta_n^1}{2T}\right) + \theta_n^2 G_n\left(\frac{2\theta_n^2}{2T}\right) \\ & + \xi_3\theta_n^3 G_n\left(\frac{2\theta_n^3}{2T}\right) + \xi_4\theta_n^4 G_n\left(\frac{2\theta_n^4}{2T}\right) + \xi_5\theta_n^5 G_n\left(\frac{2\theta_n^5}{2T}\right)\Bigg)\Bigg). \end{aligned}$$

This implies (note that G_0^{-1} and G_n are nondecreasing), for $t = 2T$,

$$E(t) \le c_{n+1}G_0^{-1}\left(\frac{c_{n+1}}{t}G_n\left(\frac{c_{n+1}}{t}\right)\right) = c_{n+1}G_{n+1}\left(\frac{c_{n+1}}{t}\right), \quad \forall t > 0,$$

where

$$c_{n+1} = \max\left\{\frac{1}{\epsilon_0}, 2c_n, 2\theta_n^1, 2\theta_n^2, 2\theta_n^3, 2\theta_n^4, 2\theta_n^5, \right.$$
$$\left. 2\left(c_n + \theta_n^1 + \theta_n^2 + \xi_3\theta_n^3 + \xi_4\theta_n^4 + \xi_5\theta_n^5\right) C_{14}\right\},$$

This proves (3.10), for $n+1$. The proof of Theorem 3.1 is completed.

4. Applications

We present in this last section certain particular applications included by the abstract system (1.1) and (1.2). Let us consider an open bounded domain $\Omega \subset \mathbb{R}^N$, where $N \in \mathbb{N}^*$, with smooth boundary Γ. In case of applications (i)–(iii), we note $H = L^2(\Omega)$ endowed with its natural inner product

$$\langle w, z\rangle = \int_\Omega w(x)z(x)dx.$$

In case of application (iv), we note $H = (L^2(\Omega))^N$ endowed with the natural inner product

$$\langle w, z\rangle = \int_\Omega \sum_{i=1}^N w_i(x)z_i(x)dx.$$

We consider the following particular coupled systems in Ω with Dirichlet and/or Neumann boundary conditions on Γ:

(i) *Wave-wave*:

$$(A, \tilde{A}, \tilde{B}) = \left(-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right), -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}\frac{\partial}{\partial x_j}\right), \tilde{b}Id\right) \tag{4.1}$$

and

$$B = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial}{\partial x_j}\right) \quad \text{or} \quad B = Id \tag{4.2}$$

with the homogeneous Dirichlet boundary conditions

$$u = 0, \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \tag{4.3}$$

and

$$v = 0, \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \tag{4.4}$$

where $a_{ij}, \tilde{a}_{ij}, b_{ij}, i, j = 1, \cdots, N$, and $\tilde{b}$ are variable coefficients depending only on the space variable and satisfying the following smoothness, symmetry, and coercivity conditions: $a_{ij}, \tilde{a}_{ij}, b_{ij} \in C^1(\bar{\Omega})$ and $\tilde{b} \in C(\bar{\Omega})$ such that

$$a_{ij}(x) = a_{ji}(x),\ \tilde{a}_{ij}(x) = \tilde{a}_{ji}(x),\ b_{ij}(x) = b_{ji}(x), \quad \forall i, j = 1, \cdots, N,\ \forall x \in \Omega \tag{4.5}$$

and there exist $d_1, d_2, d_3 > 0$ satisfying, for all $\epsilon_1, \cdots, \epsilon_N \in \mathbb{R}$ and $x \in \Omega$,

$$\sum_{i,j=1}^N a_{ij}(x)\epsilon_i\epsilon_j \geq d_1\sum_{i=1}^N \epsilon_i^2,\ \sum_{i,j=1}^N \tilde{a}_{ij}(x)\epsilon_i\epsilon_j \geq d_2\sum_{i=1}^N \epsilon_i^2,\ \sum_{i,j=1}^N b_{ij}(x)\epsilon_i\epsilon_j \geq d_3\sum_{i=1}^N \epsilon_i^2 \tag{4.6}$$

and

$$\inf_{x\in\Omega} \tilde{b}(x) > 0 \quad \text{or} \quad \sup_{x\in\Omega} \tilde{b}(x) < 0.$$

Note that (2.1), (2.2), (2.5), (3.2), and (3.4) or (3.5) are satisfied, and (2.3), (2.6), and (3.3) hold provided that g_0 and $\|\tilde{b}\|_\infty$ are small enough. On the other hand, (3.1) holds for $j_0 = 0$ if $B = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial}{\partial x_j})$, and for $j_0 = 1$ if $B = Id$. Consequently, the stability estimate (3.10) holds for $i_0 = 2$ if $B = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial}{\partial x_j})$ and $\tilde{a}_{ij} = a_{ij} - g_0 b_{ij}$. Otherwise, (3.10) holds for $i_0 = 3$.

(ii) *Petrovsky-Petrovsky*:

$$(A, \tilde{A}, \tilde{B}) = \left(a\Delta^2, \tilde{a}\Delta^2, \tilde{b}Id\right) \tag{4.7}$$

with the homogeneous Dirichlet and Neumann boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on} \quad \Gamma \times \mathbb{R}_+ \tag{4.8}$$

and

$$v = \frac{\partial v}{\partial \nu} = 0, \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \tag{4.9}$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative, $\tilde{b}$ is as in application (i), and a and $\tilde{a}$ are positive constants. Here,

$$B = \Delta^2 \quad \text{or} \quad B = Id, \tag{4.10}$$

so (3.1) holds for $j_0 = 0$ if $B = \Delta^2$, and for $j_0 = 1$ if $B = Id$. Therefore, (3.10) holds for $i_0 = 2$ if $B = \Delta^2$ and $\tilde{a} = a - g_0$. Otherwise, (3.10) holds for $i_0 = 3$.

Under the boundary conditions (4.9) and

$$u = \Delta u = 0, \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \tag{4.11}$$

we can consider $B = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial}{\partial x_j})$, where b_{ij} are as in application **(i)** (so (3.1) holds for $j_0 = 1$), and then we get (3.10) for $i_0 = 3$.

(iii) *Wave-Petrovsky*:

$$(A, \tilde{A}, \tilde{B}) = \left(-\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right), \tilde{a}\Delta^2, \tilde{b}Id\right) \tag{4.12}$$

with the boundary conditions (4.3) and (4.9), where $\tilde{a}$ is a positive constant, and a_{ij} and $\tilde{b}$ are as in application (i). The operator B can be taken as in (4.2), which implies that (3.1) holds for $j_0 = 0$ if $B = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(b_{ij}\frac{\partial}{\partial x_j})$, and for $j_0 = 1$ if $B = Id$. Then, (3.10) holds for $i_0 = 3$.

(iv) *Elasticity-elasticity*: $u = (u_1, \cdots, u_N)^T$, $v = (v_1, \cdots, v_N)^T$,

$$Au = -\frac{1}{2}\left(\sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(a_{1jkl}\left(\frac{\partial}{\partial x_k}u_l + \frac{\partial}{\partial x_l}u_k\right)\right), \cdots, \sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(a_{Njkl}\left(\frac{\partial}{\partial x_k}u_l + \frac{\partial}{\partial x_l}u_k\right)\right)\right),$$

$$\tilde{A}v = -\frac{1}{2}\left(\sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(\tilde{a}_{1jkl}\left(\frac{\partial}{\partial x_k}v_l + \frac{\partial}{\partial x_l}v_k\right)\right), \cdots, \sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(\tilde{a}_{Njkl}\left(\frac{\partial}{\partial x_k}v_l + \frac{\partial}{\partial x_l}v_k\right)\right)\right),$$

$$Bu = -\frac{1}{2}\left(\sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(b_{1jkl}\left(\frac{\partial}{\partial x_k}u_l + \frac{\partial}{\partial x_l}u_k\right)\right), \cdots, \sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(b_{Njkl}\left(\frac{\partial}{\partial x_k}u_l + \frac{\partial}{\partial x_l}u_k\right)\right)\right) \tag{4.13}$$

or

$$Bu = (b_1 u_1, \cdots, b_N u_N), \tag{4.14}$$

and

$$\tilde{B}w = (\tilde{b}_1 w_1, \cdots, \tilde{b}_N w_N)$$

with the homogeneous Dirichlet boundary conditions

$$u_i = v_i = 0, \quad \text{on} \quad \Gamma \times \mathbb{R}_+,$$

where b_i are positive constants, and a_{ijkl}, $\tilde{a}_{ijkl}$, b_{ijkl} and $\tilde{b}_i$, $i, j, k, l = 1, \cdots, N$, are variable coefficients depending only on the space variable and satisfying the following smoothness, symmetry, and coercivity conditions: a_{ijkl}, $\tilde{a}_{ijkl}$, $b_{ijkl} \in C^1(\bar{\Omega})$ and $\tilde{b}_i \in C(\bar{\Omega})$ such that, for all $i, j, k, l = 1, \cdots, N$ and $x \in \Omega$,

$$a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x), \tilde{a}_{ijkl}(x) = \tilde{a}_{jikl}(x) = \tilde{a}_{klij}(x), b_{ijkl}(x) = b_{jikl}(x) = b_{klij}(x),$$

and there exist $d_1, d_2, d_3 > 0$ satisfying, for all symmetric matrix $(\epsilon_{ij})_{ij}$ and $x \in \Omega$,

$$\sum_{i,j,k,l=1}^{N} a_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq d_1 \sum_{i,j=1}^{N} \epsilon_{ij}^2, \quad \sum_{i,j,k,l=1}^{N} \tilde{a}_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq d_2 \sum_{i,j=1}^{N},$$

$$\sum_{i,j,k,l=1}^{N} b_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geq d_3 \sum_{i,j=1}^{N} \epsilon_{ij}^2 \epsilon_{ij}^2$$

and

$$\min_{i=1,\cdots,N} \inf_{x\in\Omega} \tilde{b}_i(x) > 0 \quad \text{or} \quad \max_{i=1,\cdots,N} \sup_{x\in\Omega} \tilde{b}_i(x) < 0.$$

Condition (3.1) is satisfied for $j_0 = 0$ in case (4.13), and for $j_0 = 1$ in case (4.14). Consequently, (3.10) holds for $i_0 = 2$ provided that (4.13) holds and $\tilde{a}_{ijkl} = a_{ijkl} - g_0 b_{ijkl}$. Otherwise, (3.10) holds for $i_0 = 3$.

For more details concerning the stability (and exact controllability) of single elasticity systems with variable coefficients (depending on both space and time variables) and (internal or boundary) dampings, we refer the reader to [46] and the references therein.

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