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On the decay estimates for elasticity systems with some localized dissipations

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Abstract. We prove some precise decay estimates of the energy for non-isotropic elastodynamic systems with some localized dissipations. The damping is nonlinear and is effective only in a neighborhood of a suitable subset of the boundary, we study both degenerate and nondegenerate cases. The method of proof is direct and is based on the multiplier technique and on some specific integral inequalities.

Keywords: Decay, localized dissipation, elasticity system, multiplier method

1. Introduction and statement of the results

Let Ω be a non-empty bounded open set in $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) having a boundary Γ of class C^2 and let a_{ijkl} be functions in $W^{1,\infty}(\Omega)$ such that

$$a_{ijkl} = a_{klij} = a_{jikl} \quad \text{in } \Omega$$

(all indices run over the integers $1, 2, \ldots, n$), satisfying the ellipticity condition

$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqslant \alpha\varepsilon_{ij}\varepsilon_{ij} \quad \text{in } \Omega$$

for some fixed $\alpha > 0$ and for every symmetric tensor ε_{ij}. (We shall use the summation convention for repeated indices.)

For a given function $u = (u_1, \ldots, u_n) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$ we shall use the notations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \sigma_{ij} = a_{ijkl}\varepsilon_{kl},$$

where $u_{i,j} = \partial u_i/\partial x_j$ and $u_{j,i} = \partial u_j/\partial x_i$.

Throughout this paper, we will use the following notations. Fix a point $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$, let

$$m(x) = x - x^0, \qquad R = \|m\|_{L^\infty(\Omega)},$$

and we introduce three real Hilbert spaces H, V and W by setting

$$H = \left(L^2(\Omega)\right)^n, \qquad \|v\|_H^2 = \int_\Omega v_i v_i \,\mathrm{d}x,$$

$$V = \left(H_0^1(\Omega)\right)^n, \qquad \|v\|_V^2 = \int_\Omega \sigma_{ij}(v)\varepsilon_{ij}(v)\,\mathrm{d}x,$$

where $H_0^1(\Omega) = \{v \in H^1(\Omega)\colon v = 0 \text{ on } \Gamma\}$ (by the Korn inequality, it is clear that this expression defines a norm on V), and

$$W = (H^2(\Omega) \cap H_0^1(\Omega))^n, \qquad \|v\|_W^2 = \int_\Omega (\Delta v_i \Delta v_i + \sigma_{ij}(v)\varepsilon_{ij}(v))\,dx.$$

In this paper, we are interested in the precise decay property of the solution for elasticity systems with a localized nonlinear dissipation:

$$\begin{cases} u_i'' - \sigma_{ij,j} + l_i(x, u_i') = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u_i = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 & \text{in } \Omega, \end{cases} \tag{P1}$$

where $l_i(x, u_i') = b_i(x) g_i(u_i')$, $b_i \in L^\infty(\Omega)$, are bounded nonnegative functions and $g_i : \mathbb{R} \to \mathbb{R}$ are non-decreasing continuous functions such that $g_i(0) = 0$ and satisfying, for a constant $c' > 0$,

$$|g_i(s)| \leqslant c'(1 + |s|), \quad \text{for all } s \in \mathbb{R}.$$

The well-posedness of problem (P1) can be established by standard nonlinear semigroup theory exactly as, e.g., Guesmia [3] in the case $b_i(x) = 1$ for all $x \in \Omega$, we omit the details here. Let $(u^0, u^1) \in V \times H$ and u be the unique solution of (P1) in class $u \in C(\mathbb{R}^+; V) \cap C^1(\mathbb{R}^+; H)$, then a simple computation shows that the energy of u defined by the formula

$$E(t) = \frac{1}{2}\int_\Omega (u_i' u_i' + \sigma_{ij}\varepsilon_{ij})\,dx, \quad t \in \mathbb{R}^+, \tag{1.1}$$

satisfies

$$E'(t) = -\int_\Omega b_i u_i' g_i(u_i')\,dx \leqslant 0, \quad t \in \mathbb{R}^+; \tag{1.2}$$

hence the energy is non-increasing.

Stabilization of the wave equation using a locally distributed damping was studied by several authors, under different hypotheses. Among these, we can mention Zuazua [9], Nakao [7] and Tcheugoué Tébou [8]. The semi-group approach or differential inequalities was used by the authors to establish exponential or polynomial decay of the energy.

Zuazua [9] studied the semilinear wave equation where the damping term is linear and effective in an open nonempty subset of the domain contained the whole boundary Γ. The author considered only the nondegenerate case and he obtained exponential decay of the energy. Nakao [7] generalized these results to the nonlinear case where the damping $\rho(x, u')$ behavies like $a(x)|u'|^r u'$ with $r > -1$ and $a(x)$ is effective onlty in a neighborhood of a subset of the boundary. Nakao [7] considered also the degenerate case with a linear damping, he proved a polynomial decay of the energy for initial states belonging to

$$(H^{m+1}(\Omega) \cap H_0^1(\Omega)) \times (H^m(\Omega) \cap H_0^1(\Omega))$$

with $m \in \mathbb{N}^*$ and $2m > n$.

The method applied in these two papers is based on the multiplier technique, on some differential inequalities and on compactness–uniqueness argument to absorb lower-order terms. In the degenerate case, Nakao [7] used also the fact that the solution belongs to

$$\bigcap_{k=0}^{m} C^k(\mathbb{R}^+; H^{m+1-k}(\Omega) \cap H_0^1(\Omega)) \cap C^{m+1}(\mathbb{R}^+; L^2(\Omega)).$$

In the nonlinear case, this property of solution is no longer true in general, and then the question concerning the stabilization in the degenerate case with a nonlinear damping remains open.

In [8], Tcheugoué Tébou removed the condition $2m > n$ assumed by Nakao [7] and obtained the same decay estimates as in Nakao [7] for the linear wave equation assuming the same condition on intial states. In order to get rid of lower-order terms, Tcheugoué Tébou [8] introduced an auxiliary elliptic problem whose solution was used as multiplier, instead of the unique continuation property and the compactness argument used by Zuazua [9] and Nakao [7].

Concerning the stabilization of elasticity systems with localized dissipations, no result is known until now. In this paper, we consider nonlinear elasticity systems with a local degenerate or nondegenerate dissipation. Our results generalize and improve some of the results obtained in the papers mentioned above. We give sufficient hypotheses on the functions g_i so that we can obtain precise decay estimates of the energy. In the degenerate case, we give a positive answer to the question cited above where we impose that the initial states belong only to

$$(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega).$$

Additionally, in the linear case and for $n \geqslant 2$, we obtain a decay rate sharper than the one obtained in Nakao [7] and Tcheugoué Tébou [8].

We give now some decay estimates of the energy $E(t)$ as $t \to \infty$. Let $c_1, c_2 > 0$ and $p, r \geqslant 1$ be four fixed constants such that the functions g_i verify the condition, for all $s \in \mathbb{R}$,

$$c_1 \min\{|s|, |s|^r\} \leqslant |g_i(s)| \leqslant c_2 \max\{|s|, |s|^{1/p}\}. \tag{1.3}$$

Remark. We have many possibilities to take the functions g_i such that conditions (1.3) is satisfied, for example,

$$g_i(s) = \begin{cases} a_i|s|^{r-1}s & \text{if } |s| \leqslant 1, \\ a_i s & \text{if } |s| \geqslant 1, \end{cases}$$

where $a_i > 0$ and we take $c_1 = \min\{a_i\}$, $c_2 = \max\{a_i\}$.

As similar cases we know the following results:

1. If $b_i(x) > 0$ on Ω, then (without assumption (1.3))

$$E(t) \to 0, \quad \text{as } t \to \infty.$$

This result can be proved directly by applying the LaSalle's invariance principle.

2. If $r = p$ and $b_i(x) \geqslant \varepsilon > 0$ on Ω (we can take in this case $b_i(x) = 1$ for all $x \in \Omega$ without loss of generality), then

$$E(t) \leqslant ct^{-2/(p-1)}, \quad t > 0, \text{ if } p > 1, \tag{1.4}$$

and

$$E(t) \leqslant E(0)\mathrm{e}^{1-\omega_0 t}, \quad t > 0, \text{ if } p = 1, \tag{1.5}$$

where $c > 0$ depends on $E(0)$ and $\omega_0 > 0$ is independent of the initial data. The estimates (1.4) and (1.5) were proved in Guesmia [3].

Let us consider a more delicate case. For this, we introduce

$$\Gamma_+ = \{x \in \Gamma\colon\ m(x).\nu(x) > 0\},$$

where ν denotes the outward unit normal vector to Γ. We suppose that there exists a neighborhood ω of Γ_+ (which means that ω is the intersection of Ω and a neighborhood of Γ_+ in $\mathbb{R}^n$), such that

$$b_i(x) \geqslant b_0 > 0 \quad \text{on } \omega, \tag{1.6}$$

or

$$b_i(x) > 0 \quad \text{on } \omega \quad \text{and} \quad \int_\omega b_i^{-p'}(x)\,\mathrm{d}x < \infty \tag{1.7}$$

for some $p' > 0$ such that

$$p'(p-1) > 2 \quad \text{and} \quad p'(p-1) \geqslant n. \tag{1.8}$$

We impose also on a_{ijkl} the following condition: there exists a constant $\gamma > 0$ such that

$$(2a_{ijkl} - m_p\partial_p a_{ijkl})\varepsilon_{ij}\varepsilon_{kl} \geqslant \gamma a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \quad \text{in } \Omega \tag{1.9}$$

for every symmetric tensor ε_{ij}, where $\partial_p a_{ijkl} = \partial a_{ijkl}/\partial x_p$. We can get condition (1.9) by taking the derivative of a_{ijkl} small with respect to a_{ijkl}. If $a_{ijkl} =$ const, then of course we have $\gamma = 2$.

Our result read as follows:

Theorem 1.1. *Assume* (1.9).

1. *Under the hypotheses* (1.6) *and* (1.3) *with* $r = p$ *we have, for every* $(u^0, u^1) \in V \times H$, *the energy* E *verifies the estimates* (1.4) *and* (1.5).
2. *Under the hypotheses* (1.7), (1.8) *and* (1.3) *with* $r = 1$ *we have, for every* $(u^0, u^1) \in W \times V$, *the energy* E *verifies the estimate* (1.4) *with a constant* $c > 0$ *depending on the solution* u.

Remarks. 1. Assumption (1.3) implies that the functions g_i are not bounded. In [3], we have proved some decay estimates in the case $b_i(x) = 1$ for all $x \in \Omega$ even if the functions g_i are bounded. These estimates seem to be proved in the case of (1.6). In order to keep this paper not too long, we only consider the case of (1.3).

2. The decay estimates will be proved under the restrictive condition (1.9). This condition has been assumed in Guesmia [2] to obtain the observability inequalities for (P1) with $b_i \equiv 0$. The general case remains open.

3. In the nondegenerate case (1.6), stabilization results can be obtained without any growth condition on the functions g_i at the origin; that is, we consider condition (1.3) only for $s \in \mathbb{R}$ such that $|s| \geqslant 1$, by using microlocal estimates as was done in [5] for the wave equations. Unlike our results, due to an indirect compactness–uniqueness argument, the decay rates will not be explicit in this case; they are usually described by a nonlinear dissipative ordinary equation. Similarly, applying a method developed in [6], we can remove condition (1.3) for $s \in \mathbb{R}$ such that $|s| < 1$ and we obtain a decay rate of this form:

$$E(t) \leqslant c\left(G^{-1}\left(\frac{1}{t}\right)\right)^2, \quad \forall t > 0,$$

where $G(s) = sg(s)$ for all $s \in \mathbb{R}$ and c is a positive constant. To do so we need additional conditions like: the fuctions g_i are of class C^1 and they are Lipschitz or they are odd.

On the other hand, these two methods mentioned above seem to be not applicable in the degenerate case (1.7) due to the degenerescence of b_i. In order to make the paper not too long we do not study these questions here.

2. Proof of Theorem 1.1

The proof of the decay estimates is given by combining the ideas in Guesmia [2,3], Tcheugoué Tébou [8] and Zuazua [9]. We are going to prove that the energy satisfies the estimate

$$\int_S^{\infty} E^{(p+1)/2}(t)\,\mathrm{d}t \leqslant cE(S) \tag{2.1}$$

for all $0 \leqslant S < +\infty$. Here and in the sequel we shall denote by c diverse positive constants. We recall that if a nonnegative and nonincreasing function $E:\mathbb{R}^+ \to \mathbb{R}^+$ satisfies the estimate (2.1), then it also satisfies (1.4) and (1.5) (see Guesmia [3, Proposition 3.7]). Then Theorem 1.3 will be proved if we establish inequality (2.1).

Remark. All computations that follow will be justified for strong solution; that is $(u^0, u^1) \in W \times V$ (see Guesmia [3, Theorem 1.2]). Since the constants c and ω_0 in (1.4) and (1.5), respectively, will not depend on u in the nondegenerate case (1.6), once the estimates (1.4) and (1.5) will be estabilished for strong solutions, they will be also satisfied for all weak solutions by an easy density argument. In the degenerate case (1.7), this is not possible because the constant c in (1.4) will depend on u; more precisely, $c = c(\|u'\|_{L^\infty(\mathbb{R}^+;V)})$, hence in this case, the estimate (1.4) is proved only for strong solutions. Concerning the weak solutions, the estimate (1.4) seems to be not true in general because they do not have the property (2.20) below.

We start this section by giving an explicit formula satisfied by energy. Integrating the estimate (1.2) over $[S,T]$, where $0 \leqslant S < T < \infty$, we obtain easily that

$$\int_S^T \int_\Omega b_i u_i' g_i(u_i')\,dx\,dt = E(S) - E(T) \leqslant E(S). \tag{2.2}$$

In order to prove (2.1) we proceed in several steps.

Step 1. We multiply Eq. (P1) by $2E^{(p-1)/2}(t) h_m u_{i,m}$ with a vector field $h \in (W^{1,\infty}(\Omega))^n$. We easily obtain the identity (as in the proof of identity (2.4) in Guesmia [2])

$$\begin{aligned}
&\int_S^T E^{(p-1)/2}(t) \int_\Omega \big(2h_{m,j}\sigma_{ij}u_{i,m} + (\operatorname{div} h)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij})\big)\,dx\,dt \\
&\quad = \int_S^T E^{(p-1)/2}(t) \int_\Gamma (h.\nu)\sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt + \int_S^T E^{(p-1)/2}(t) \int_\Omega h_m(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}\,dx\,dt \\
&\qquad - 2\int_S^T E^{(p-1)/2}(t) \int_\Omega h_m u_{i,m} b_i g_i(u_i')\,dx\,dt + \left[2E^{(p-1)/2}(t)\int_\Omega h_m u_{i,m} u_i'\,dx\right]_T^S \\
&\qquad + (p-1)\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega h_m u_{i,m} u_i'\,dx\,dt.
\end{aligned} \tag{2.3}$$

Using the definition of energy and the Hölder and Korn inequalities, we find that

$$\left|2E^{(p-1)/2}(t)\int_\Omega h_m u_{i,m} u_i'\,dx\right| \leqslant cE^{(p+1)/2}(t)$$

and

$$\left|(p-1)E^{(p-3)/2}(t)E'(t)\int_\Omega h_m u_{i,m} u_i'\,dx\right| \leqslant cE^{(p-1)/2}(t)(-E'(t)).$$

Using the nonincreasing of energy, we deduce that the last two terms of (2.3) can be easily majorized by $cE^{(p+1)/2}(S)$. Then, applying identity (2.3) with $h = m$ and using (1.9) and the definition of Γ_+, we conclude the estimate

$$\begin{aligned}
&\int_S^T E^{(p-1)/2}(t) \int_\Omega \big((\gamma - n)\sigma_{ij}\varepsilon_{ij} + n u_i'u_i'\big)\,dx\,dt \\
&\quad \leqslant -2\int_S^T E^{(p-1)/2}(t)\int_\Omega m_k u_{i,k} b_i g_i(u_i')\,dx\,dt + cE^{(p+1)/2}(S) \\
&\qquad + R\int_S^T E^{(p-1)/2}(t)\int_{\Gamma_+} \sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt.
\end{aligned} \tag{2.4}$$

We take now in (2.3) a function $h \in (W^{1,\infty}(\Omega))^n$ such that

$$h = \nu \quad \text{on } \Gamma_+, \qquad h.\nu \geqslant 0 \quad \text{on } \Gamma \quad \text{and} \quad h = 0 \quad \text{on } \widetilde{\omega}^c,$$

where $\widetilde{\omega}$ is another neighborhood of Γ_+ strictly contained in ω (see Zuazua [9] and the references cited there for the construction of this vector field), we deduce

$$
\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\int_{\Gamma_+}\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad=\int_S^T E^{(p-1)/2}(t)\int_{\Gamma_+}(h.\nu)\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}\Gamma\,\mathrm{d}t\leqslant\int_S^T E^{(p-1)/2}(t)\int_{\Gamma}(h.\nu)\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad\leqslant c\int_S^T E^{(p-1)/2}(t)\int_{\widetilde{\omega}}(\sigma_{ij}\varepsilon_{ij}+u_i'u_i')\,\mathrm{d}x\,\mathrm{d}t+cE^{(p+1)/2}(S)\\
&\qquad+2\int_S^T E^{(p-1)/2}(t)\int_{\widetilde{\omega}}h_m u_{i,m}b_i g_i(u_i')\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.5)
\end{aligned}
$$

On the other hand, we have, for any $\varepsilon>0$ (note that b_i is bounded),

$$
\begin{aligned}
&\left|2\int_S^T E^{(p-1)/2}(t)\int_{\Omega}m_k u_{i,k}b_i g_i(u_i')\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad\leqslant\frac{\varepsilon}{2}\int_S^T E^{(p+1)/2}(t)\,\mathrm{d}t+c\int_S^T E^{(p-1)/2}(t)\int_{\Omega}b_i g_i^2(u_i')\,\mathrm{d}x\,\mathrm{d}t,\\
&\left|2\int_S^T E^{(p-1)/2}(t)\int_{\widetilde{\omega}}h_m u_{i,m}b_i g_i(u_i')\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad\leqslant\frac{\varepsilon}{2R}\int_S^T E^{(p+1)/2}(t)\,\mathrm{d}t+c\int_S^T E^{(p-1)/2}(t)\int_{\Omega}b_i g_i^2(u_i')\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Combining the above inequalities for $\varepsilon\in]0,1[$ with (2.4) and (2.5) we deduce

$$
\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\int_{\Omega}\big((\gamma-n)\sigma_{ij}\varepsilon_{ij}+nu_i'u_i'\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant cE^{(p+1)/2}(S)+c\int_S^T E^{(p-1)/2}(t)\int_{\Omega}b_i g_i^2(u_i')\,\mathrm{d}x\,\mathrm{d}t+\varepsilon\int_S^T E^{(p+1)/2}(t)\,\mathrm{d}t\\
&\qquad+c\int_S^T E^{(p-1)/2}(t)\int_{\widetilde{\omega}}(\sigma_{ij}\varepsilon_{ij}+u_i'u_i')\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.6)
\end{aligned}
$$

Step 2. We now estimate the quantity $\int_S^T E^{(p-1)/2}(t)\int_{\widetilde{\omega}}\sigma_{ij}\varepsilon_{ij}\,\mathrm{d}x\,\mathrm{d}t$. We multiply Eq. (P1) by $E^{(p-1)/2}(t)\eta(x)u_i$ with $\eta\in W^{1,\infty}(\Omega)$. Integrating by parts we obtain the following identity:

$$
\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\int_{\Omega}\eta(\sigma_{ij}\varepsilon_{ij}-u_i'u_i')\,\mathrm{d}x\,\mathrm{d}t+\int_S^T E^{(p-1)/2}(t)\int_{\Omega}(\partial_j\eta)\sigma_{ij}u_i\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\int_S^T E^{(p-1)/2}(t)\int_{\Omega}\eta b_i g_i(u_i')u_i\,\mathrm{d}x\,\mathrm{d}t+\left[E^{(p-1)/2}(t)\int_{\Omega}\eta u_i u_i'\,\mathrm{d}x\right]_T^S\\
&\qquad+\frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_{\Omega}\eta u_i u_i'\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.7)
\end{aligned}
$$

Using the defintion of energy, the last two terms of this identity can be easily majorized by $cE^{(p+1)/2}(S)$. Then, applying identity (2.7) with $\eta = n - \gamma/2$ we get

$$\int_S^T E^{(p-1)/2}(t)\int_\Omega \left(n-\frac{\gamma}{2}\right)(\sigma_{ij}\varepsilon_{ij} - u_i' u_i')\,dx\,dt$$
$$\leqslant cE^{(p+1)/2}(S) - \left(n-\frac{\gamma}{2}\right)\int_S^T E^{(p-1)/2}(t)\int_\Omega b_i g_i(u_i')u_i\,dx\,dt. \tag{2.8}$$

We take now in (2.7) a function $\eta \in W^{1,\infty}(\Omega)$ such that

$$\eta = 1 \quad \text{in } \widetilde{\omega}, \qquad 0 \leqslant \eta \leqslant 1 \quad \text{in } \Omega \quad \text{and} \quad \eta = 0 \quad \text{in } \omega^c$$

(see Zuazua [9] and the references cited there for the construction of this function), we deduce

$$\int_S^T E^{(p-1)/2}(t)\int_{\widetilde{\omega}} \sigma_{ij}\varepsilon_{ij}\,dx\,dt \leqslant \int_S^T E^{(p-1)/2}(t)\int_\Omega \eta\sigma_{ij}\varepsilon_{ij}\,dx\,dt$$
$$\leqslant cE^{(p+1)/2}(S) - \int_S^T E^{(p-1)/2}(t)\int_\omega (\partial_j\eta)\sigma_{ij}u_i\,dx\,dt$$
$$+ \int_S^T E^{(p-1)/2}(t)\int_\omega \left(u_i'u_i' + b_i|g_i(u_i')u_i|\right)dx\,dt. \tag{2.9}$$

We have

$$\int_S^T E^{(p-1)/2}(t)\int_\omega b_i|g_i(u_i')u_i|\,dx\,dt \leqslant c\int_S^T E^{(p-1)/2}(t)\int_\omega \left(u_iu_i + b_ig_i^2(u_i')\right)dx\,dt,$$
$$\left|\int_S^T E^{(p-1)/2}(t)\int_\omega (\partial_j\eta)\sigma_{ij}u_i\,dx\,dt\right| \leqslant \varepsilon\int_S^T E^{(p+1)/2}(t)\,dt + c\int_S^T E^{(p-1)/2}(t)\int_\omega u_iu_i\,dx\,dt$$

and

$$\left|\left(n-\frac{\gamma}{2}\right)\int_S^T E^{(p-1)/2}(t)\int_\Omega b_ig_i(u_i')u_i\,dx\,dt\right|$$
$$\leqslant \varepsilon\int_S^T E^{(p+1)/2}(t)\,dt + c\int_S^T E^{(p-1)/2}(t)\int_\Omega b_ig_i^2(u_i')\,dx\,dt$$

for any $\varepsilon > 0$. Combining the above inequalities with (2.6), (2.8) and (2.9) where ε is taken small enough, we deduce (note that $\widetilde{\omega} \subset \omega$)

$$\int_S^T E^{(p+1)/2}(t)\,dt \leqslant cE^{(p+1)/2}(S) + c\int_S^T E^{(p-1)/2}(t)\int_\Omega b_ig_i^2(u_i')\,dx\,dt$$
$$+ c\int_S^T E^{(p-1)/2}(t)\int_\omega u_i'u_i'\,dx\,dt + c\int_S^T E^{(p-1)/2}(t)\int_\omega u_iu_i\,dx\,dt. \tag{2.10}$$

Step 3. To absorb the last integral of (2.10), we adapt to our system, the method given by Tcheugoué Tébou [8] in the study of the linear wave equation. We prove then the following lemma:

Lemma 2.1. *For any given* $\varepsilon > 0$, *we have*

$$\int_S^T E^{(p-1)/2}(t) \int_\omega u_i u_i \, dx \, dt \leqslant \varepsilon \int_S^T E^{(p+1)/2}(t) \, dt + cE^{(p+1)/2}(S) + c \int_S^T E^{(p-1)/2}(t) \left(\int_\omega u_i' u_i' \, dx + \int_\Omega b_i g_i^2(u_i') \, dx \right) dt. \tag{2.11}$$

Proof. For every $t \in \mathbb{R}^+$ let us denote by $z(t)$ the solution of the problem

$$\begin{cases} -\sigma_{ij,j}(z) = \chi(\omega) u_i & \text{in } \Omega, \\ z_i = 0 & \text{on } \Gamma, \end{cases}$$

where $\chi(\omega)$ is the characteristic function of ω. Then we have

$$\int_\omega u_i z_i \, dx = \int_\Omega \chi(\omega) u_i z_i \, dx = -\int_\Omega \sigma_{ij,j}(z) z_i \, dx = \int_\Omega \sigma_{ij}(z) \varepsilon_{ij}(z) \, dx = \|z\|_V^2.$$

Since

$$\|z\|_V^2 \leqslant c \int_\omega u_i u_i \, dx \tag{2.12}$$

(the constant c is not depending on u). Applying (2.12) with u' instead of u we also obtain

$$\|z'\|_V^2 \leqslant c \int_\omega u_i' u_i' \, dx. \tag{2.13}$$

Let us also observe that

$$\int_\omega u_i u_i \, dx = \int_\Omega \chi(\omega) u_i u_i \, dx = -\int_\Omega \sigma_{ij,j}(z) u_i \, dx = \int_\Omega \sigma_{ij}(z) \varepsilon_{ij}(u) \, dx. \tag{2.14}$$

Multiplying Eq. (P1) by $z_i E^{(p-1)/2}(t)$, integrating by parts and using (2.14) we get

$$\int_S^T E^{(p-1)/2}(t) \int_\omega u_i u_i \, dx \, dt = \int_S^T E^{(p-1)/2}(t) \int_\Omega \left(z_i' u_i' - z_i b_i g_i(u_i') \right) dx \, dt + \left[E^{(p-1)/2}(t) \int_\Omega z_i u_i' \, dx \right]_T^S + \frac{p-1}{2} \int_S^T E^{(p-3)/2}(t) E'(t) \int_\Omega z_i u_i' \, dx \, dt. \tag{2.15}$$

Using (2.12), the last two terms of (2.15) can be majorized by $cE^{(p+1)/2}(S)$. On the other hand, using (2.12), (2.13) and the Young inequality,

$$\begin{aligned} \int_S^T E^{(p-1)/2}(t) \int_\Omega z_i' u_i' \, dx \, dt &\leqslant \varepsilon' \int_S^T E^{(p-1)/2}(t) \int_\Omega u_i' u_i' \, dx \, dt + c \int_S^T E^{(p-1)/2}(t) \int_\Omega z_i' z_i' \, dx \, dt \\ &\leqslant \varepsilon' \int_S^T E^{(p-1)/2}(t) \int_\Omega u_i' u_i' \, dx \, dt + c \int_S^T E^{(p-1)/2}(t) \int_\omega u_i' u_i' \, dx \, dt, \end{aligned}$$

$$-\int_S^T E^{(p-1)/2}(t)\int_\Omega z_i b_i g_i(u_i')\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_S^T E^{(p-1)/2}(t)\left(\varepsilon'\int_\Omega z_i z_i\,\mathrm{d}x\,\mathrm{d}t + c\int_\Omega b_i g_i^2(u_i')\,\mathrm{d}x\right)\mathrm{d}t$$

$$\leqslant \varepsilon'\int_S^T E^{(p-1)/2}(t)\int_\omega u_i u_i\,\mathrm{d}x\,\mathrm{d}t + c\int_S^T E^{(p-1)/2}(t)\int_\Omega b_i g_i^2(u_i')\,\mathrm{d}x\,\mathrm{d}t$$

for $\varepsilon' \in]0,1[$. Substituting these two inequalities into (2.15) and taking $\varepsilon = 2\varepsilon'/(1-\varepsilon')$ we derive (2.11).

Now, we take $\varepsilon > 0$ small enough, it follows from (2.10) and (2.11) that

$$\int_S^T E^{(p+1)/2}(t)\,\mathrm{d}t \leqslant cE^{(p+1)/2}(S) + c\int_S^T E^{(p-1)/2}(t)\left(\int_\omega u_i'u_i'\,\mathrm{d}x + \int_\Omega b_i g_i^2(u_i')\,\mathrm{d}x\right)\mathrm{d}t. \quad (2.16)$$

Step 4. To estimate the last integral of (2.16), using assumptions (1.6)–(1.8) and (1.3), we prove the following lemma:

Lemma 2.2. *We have, for all $t \in \mathbb{R}^+$,*

$$\int_\Omega b_i g_i^2(u_i')\,\mathrm{d}x \leqslant -cE'(t) + c\big(-E'(t)\big)^{2/(p+1)} \quad (2.17)$$

and

$$\int_\omega u_i'u_i'\,\mathrm{d}x \leqslant -cE'(t) + c\big(-E'(t)\big)^{2/(p+1)}. \quad (2.18)$$

Proof. Fix $t \geqslant 0$ arbitrarily and set (as in Guesmia [3])

$$\Omega_i^- = \{x \in \Omega\colon |u_i'(x)| \leqslant 1\}, \qquad \Omega_i^+ = \{x \in \Omega\colon |u_i'(x)| > 1\}.$$

Using the growth assumption (1.3) and the Hölder inequality, we have (note that b_i is bounded)

$$\int_{\Omega_i^-} b_i g_i^2(u_i')\,\mathrm{d}x \leqslant c\int_{\Omega_i^-}\big(b_i u_i' g_i(u_i')\big)^{2/(p+1)}\,\mathrm{d}x \leqslant c\left(\int_{\Omega_i^-} b_i u_i' g_i(u_i')\,\mathrm{d}x\right)^{2/(p+1)}$$

$$\leqslant c\left(\int_\Omega b_i u_i' g_i(u_i')\,\mathrm{d}x\right)^{2/(p+1)} \leqslant c\big(-E'(t)\big)^{2/(p+1)}$$

(we applied (1.2) in the last step) and

$$\int_{\Omega_i^+} b_i g_i^2(u_i')\,\mathrm{d}x \leqslant c\int_{\Omega_i^+} b_i u_i' g_i(u_i')\,\mathrm{d}x \leqslant -cE'(t).$$

Taking their sum we obtain (2.17).

To prove (2.18), we use (1.6) or (1.7) and (1.8).

Assume (1.6). Then (note that, in this case, we have assumed that $r = p$)

$$\int_\omega u_i'u_i'\,\mathrm{d}x \leqslant \frac{1}{b_0}\int_\omega b_i u_i'u_i'\,\mathrm{d}x \leqslant \frac{1}{b_0}\int_\Omega b_i u_i'u_i'\,\mathrm{d}x,$$

hence, using assumption (1.3), we may prove in the same way the estimate (2.18).

Assume (1.7) and (1.8). We have ($r = 1$)

$$c_1|x| \leqslant |g_i(x)|, \quad \text{for all } x \in \mathbb{R}. \tag{2.19}$$

Because the initial states $(u^0, u^1) \in W \times V$, then the solution u of (P1) satisfies (see Guesmia [3, Theorem 1.2])

$$u' \in L^\infty(\mathbb{R}^+; V). \tag{2.20}$$

Put $q = 1 + 4/(p'(p-1) - 2)$ (note that $q \in [1, \infty[$ and $(n-2)q \leqslant n+2$). Then, applying the Hölder inequality and using (2.19), (2.20) and the injection $V \subset (L^{q+1}(\Omega))^n$, we get

$$\begin{aligned}
\int_\omega u_i' u_i' \,dx &= \int_\omega |u_i'|^{2(p-1)/(p+1)} |u_i'|^{4/(p+1)} \,dx \\
&\leqslant c\left(\int_\omega |u_i'|^{q+1} \,dx\right)^{\frac{2(p-1)}{(q+1)(p+1)}} \left(\int_\omega |u_i'|^{\frac{4(q+1)}{(p+1)(q+1)-2(p-1)}} \,dx\right)^{1-\frac{2(p-1)}{(p+1)(q+1)}} \\
&\leqslant c\|u'(t)\|_V^{2(p-1)/(p+1)} \left(\int_\omega b_i^{-p'/(p'+1)} b_i^{p'/(p'+1)} |u_i'|^{\frac{4(q+1)}{(p+1)(q+1)-2(p-1)}} \,dx\right)^{1-\frac{2(p-1)}{(p+1)(q+1)}} \\
&\leqslant c\left[\left(\int_\omega b_i^{-p'} \,dx\right)^{1/(p'+1)} \left(\int_\omega b_i |u_i'|^{(p'+1)/p' \frac{4(q+1)}{(p+1)(q+1)-2(p-1)}} \,dx\right)^{p'/(p'+1)}\right]^{1-\frac{2(p-1)}{(p+1)(q+1)}} \\
&\leqslant c\left(\int_\Omega b_i u_i' g_i(u_i') \,dx\right)^{2/(p+1)} \leqslant c(-E'(t))^{2/(p+1)};
\end{aligned}$$

property (1.2) is used in the last step and note that

$$\frac{p'+1}{p'} \frac{4(q+1)}{(p+1)(q+1) - 2(p-1)} = 2, \qquad \frac{p'}{p'+1}\left(1 - \frac{2(p-1)}{(p+1)(q+1)}\right) = \frac{2}{p+1}.$$

Then (2.18) follows.

Substituting the estimates (2.17) and (2.18) into the right-hand side of (2.16), we obtain that

$$\begin{aligned}
\int_S^T E(t)^{(p+1)/2} \,dt &\leqslant cE(S)^{(p+1)/2} + c\int_S^T \left(-E(t)^{(p-1)/2} E'(t) + E(t)^{(p-1)/2}(-E'(t))^{2/(p+1)}\right) dt \\
&\leqslant cE(S)^{(p+1)/2} + c\int_S^T E(t)^{(p-1)/2}(-E'(t))^{2/(p+1)} \,dt.
\end{aligned}$$

Using the Young inequality, for any fixed $\varepsilon > 0$ we have

$$E(t)^{(p-1)/2}(-E'(t))^{2/(p+1)} \leqslant \varepsilon E(t)^{(p+1)/2} + c\varepsilon^{(1-p)/2}(-E'(t)).$$

Therefore, using the nonincreasing of energy,

$$(1-\varepsilon)\int_S^T E(t)^{(p+1)/2}\,\mathrm{d}t \leqslant cE(S)^{(p+1)/2} + c\varepsilon^{(1-p)/2}\int_S^T \bigl(-E'(t)\bigr)\,\mathrm{d}t$$
$$\leqslant c\bigl(1+\varepsilon^{(1-p)/2}\bigr)\bigl(1+E(S)^{(p-1)/2}\bigr)E(S) \leqslant c\bigl(1+\varepsilon^{(1-p)/2}\bigr)\bigl(1+E(0)^{(p-1)/2}\bigr)E(S);$$

choosing $0<\varepsilon<1$ and letting T go to infinity, the desired estimate (2.1) follows.

Remarks. 1. If the functions g_i are linear, that is, $g_i(s) = d_i s$ for all $s\in\mathbb{R}$ with $d_i > 0$, then condition (1.3) is satisfied for all $p\in[1,\infty[$. Then we obtain an exponential decay in the nondegenerate case (1.6). And in the degenerate case (1.7) we obtain, taking $p = n/p' + 1$ if $n\geqslant 3$ and $p = 2(1+\varepsilon)/p' + 1$ if $n = 1,2$ with $\varepsilon\in]0,1[$ (then (1.8) is satisfied),

$$E(t)\leqslant ct^{-2p'/n},\quad t>0,\ \text{if } n\geqslant 3, \tag{2.21}$$

and

$$E(t)\leqslant ct^{-p'/(1+\varepsilon)},\quad t>0,\ \text{if } n=1,2. \tag{2.22}$$

For $n\geqslant 2$, (2.21) and (2.22) give a decay rate sharper than the one obtained by Nakao [7] and Tcheugoué Tébou [8] for the linear wave equation. Additionally, no conditions are required to be imposed on the degenerescence of b_i unlike the restrictions assumed in [8].

2. We consider the non-isotropic elastodynamic system with potential of type

$$\begin{cases} u_i'' - \sigma_{ij,j} + q_i u_i + l_i(x,u_i') = 0 & \text{in } \Omega\times\mathbb{R}^+,\\ u_i = 0 \quad \text{on } \Gamma\times\mathbb{R}^+, & \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 & \text{in } \Omega, \end{cases} \tag{P2}$$

where $q_i\in L^\infty(\Omega)$ are bounded nonnegative functions. If $\max_i\{\|q_i\|_{L^\infty(\Omega)}\}$ is small enough, then results analogous to Theoerem 1.1 can be obtained using the method developed above where the energy of (P2) is given by

$$E(t) = \frac{1}{2}\int_\Omega \bigl(u_i'u_i' + \sigma_{ij}\varepsilon_{ij} + q_i u_i^2\bigr)\,\mathrm{d}x,\quad t\in\mathbb{R}^+.$$

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